# Geometry in CT Reconstruction

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1 Introduction to CT Reconstruction

The key to CT reconstruction is a characterization of how X-rays are absorbed by material. Since X-rays are electromagnetic waves, we consider how the latter are absorbed as a function of the properties of the material through which the light is traveling, as studied in the field of optics. This absorption is governed by the Beer Lambert law, also alternatively known as the Beer’s law or the Lambert Beer law or the Beer-Lambert-Bouguer, Figure xx. In short, this law states that the intensity of light passed through a slab of length $l$ drops exponentially with $l$, i.e.,

$$I(l) = I_0 e^{-\alpha l},$$

where $I_0$ is the original intensity of the light, etc...

Figure 1: An illustration of the BeerLambert law in optics

1.1 Linear Attenuation Coefficient

Consider an X-ray beam going through some material. The probability that a photon gets removed from a beam (either absorbed or scattered) depends on (i) the energy of the photon, and (ii) the nature of the material through which the beam passes. Define the linear attenuation coefficient $\mu_e^t$ of tissue $t$ at energy $e$ as

$$\mu_e^t = -\ln(p_e),$$

where $p_e$ is the probability that a photon of energy $e$ transmits through a uniform slab of unit length of tissue $t$ in a direction perpendicular to the face of the slab. For example, the linear attenuation coefficient of water at 73 kev is 0.19 $cm^{-1}$. 

3
A general model of X-ray absorption is based on an independence assumption: The extent of X-ray absorption by the material by any portion of space is independent of that corresponding to the complementary space. This independence is essentially what gives rise to the exponential decay in X-ray intensity. The following discussion is adapted from [11]. Specifically, consider a slab of non-uniform material and denote the linear attenuation coefficient as a function of the coordinate $x$ along the space as $\mu_e(x)$. Let $p_e(x)$ be the probability of absorption in a slab of width $dx$ at $x$, i.e., the probability that a photon of energy $e$ is transmitted as far as $x$. Note that $p_e(x)$ is a monotonically decreasing function of $x$. Now, consider an infinitesimal section $dx$ of the slab from $x$ to $x + dx$. Let $q_e(x, dx)$ denote the probability that a photon of energy $e$ which has reached $x$ will not be transmitted beyond $x + dx$. It is clear, therefore, that the probability of a photon reaching $x + dx$ is the probability of it reaching $x$ and being transmitted through the $dx$ slab:

$$p_e(x + dx) = p_e(x)(1 - q_e(x, dx)),$$

or

$$q_e(x, dx) = -\frac{p_e(x + dx) - p_e(x)}{p_e(x)}.$$

Now let

$$\mu_e(x) = \lim_{dx \to 0} q_e(x, dx) = -\frac{p_e'(x)}{p_e(x)} = -\frac{d(\ln(p_e(x)))}{dx}.$$

Observe that

$$\int_0^A \mu_e(x)dx = -\int_0^A -\frac{d(\ln(p_e(x)))}{dx}dx = -\ln(p_e(x))|_0^A = -\ln(p_e(A)) + \ln(1) = -\ln(p_e(A)).$$

Now, for a uniform slab of unit length, i.e. where $\mu_e(x) = \mu_e$ and $A = 1$, we have

$$\mu_e = \int_0^1 \mu_e(x)dx = -\ln(p_e(1)).$$

Comparing this to Equation(2), it is clear that $\mu_e = \mu_e^t$. Since $q_e$ as a probability is dimensionless, $\mu_e$ has dimension of $\text{length}^{-1}$ and so does $\mu_e^t$.

In actual practice, since calibration is required to make measurements uniform across detection, the actual linear attenuation coefficient $\mu_e^t$ is measured relative to $\mu_e^c$, the linear attenuation coefficient of the calibration material. Define the relative linear attenuation coefficient as

$$\mu_e^{t,a} = \mu_e^t - \mu_e^a.$$
Note that line integrals of these quantities are simply related:

\[ \int_0^A \mu_e^{t,a}(x)dx = \int_0^A (\mu_e^{t}(x) - \mu_e^{a}(x))dx \]
\[ = \int_0^A \mu_e^{t}(x)dx - \int_0^A \mu_e^{a}(x)dx \]
\[ = -\ln(p_e^{t}(A)) + \ln(p_e^{a}(A)) \]
\[ = -\ln\left(\frac{p_e^{t}(A)}{p_e^{a}(A)}\right). \]

The *CT number* is defined as

\[ H = 1000 \frac{\mu_e^{t} - \mu_e^{water}}{\mu_e^{water}}, \tag{3} \]

which is scaled so that the CT number of water is 0 and the CT number of air is -1000. CT numbers are expressed in Hounsfield units\(^1\).

<table>
<thead>
<tr>
<th>Tissue</th>
<th>CT Number (H)</th>
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<tr>
<td>Lung</td>
<td>-300</td>
</tr>
<tr>
<td>Fat</td>
<td>-90</td>
</tr>
<tr>
<td>White Matter</td>
<td>30</td>
</tr>
<tr>
<td>Gray Matter</td>
<td>40</td>
</tr>
<tr>
<td>Muscle</td>
<td>50</td>
</tr>
<tr>
<td>Trabecular Bone</td>
<td>300-500</td>
</tr>
<tr>
<td>Cortical Bone</td>
<td>600-1,000</td>
</tr>
</tbody>
</table>

1.2 Linear Tomography

Linear Tomography is a technique used to emphasize certain depths in patients before computers could be used to do this. Basically, with the patient on the table, the X-ray source and film were moved in opposite directions, each at a fixed rate. Transaxial Tomography essentially accomplishes the same idea by rotating both the patient and the film at the same speed.

The basic idea is that the images of two points A and B (both at depth \( h \) with respect to the source), \( \vec{A} \) and \( \vec{B} \), have a distance \( |\vec{A} - \vec{B}| = \frac{H_e}{\mu_e} |AB| \),

\(^1\)[From Wiki] Sir Godfrey Newbold Hounsfield CBE, FRS, (28 August 1919 – 12 August 2004) was an English electrical engineer who shared the 1979 Nobel Prize for Physiology or Medicine with Allan McLeod Cormack for his part in developing the diagnostic technique of X-ray computed tomography (CT).
regardless of the position of S, as long as the depth \( h \) is fixed and correspondence is maintained. Figure(2) shows the process of emphasizing the plane at depth \( h \) from the source. Thus, the film and the source can be moved simultaneously so that the image of A on the film in the new position of the film, \( \hat{A} \), corresponds exactly to \( A \). Since \( |\hat{A}B| = |AB| = \frac{H}{h}|AB| \), all points at the same depth align with their corresponding points. For points at other depths, however, there is no correspondence and each point \( C \) contributes to points in an extended range. Thus, the effect of these points is to add more blurring to the film, but at the same time isolating the contribution of points at depth \( h \) since they add up in the same spot on the film. Linear and transaxial tomography motivate the idea of backprojection as a way of finding the linear attenuation coefficient at each point. We first develop some formalities.

1.3 Radon Transform

Let \( \mu(x, y) \) represent the distribution of the X-ray absorption/attenuation coefficients in Hounsfield units. This is our unknown. We will assume throughout this document that \( \mu \) is continuous and bounded, and that it vanishes outside the region of interest (scanning area). The Radon Transform, \( R_\mu \) is a map from \( \mu(x, y) \) to a 1-D function \( \rho_\theta(\xi) \) where \( \theta \) is the angle along lines of X-ray projection and \( \xi \) is the variable orthogonal to these lines (Figure 3), i.e.,

\[
\rho_\theta(\xi) = \int_{-\infty}^{\infty} \mu(x(\xi, \eta), y(\xi, \eta))d\eta = \int_{-\infty}^{\infty} \mu(\xi \cos \theta - \eta \sin \theta, \xi \sin \theta + \eta \cos \theta)d\eta.
\]
We also often use $R_\mu(\xi, \theta)$ or $\rho(\xi, \theta)$ to denote the Radon Transform over a range of angles.

Figure 3: (a) The Radon Transform is the line integral of $\mu$, where each line represents the X-ray that goes from source to sensor. The line can be expressed as $\xi = \xi_0$ in a rotated coordinate system in (b).

Ideally, these measurements need to be made only for $\theta \in [0, \pi)$ since the entire $[0, 2\pi)$ range can be covered using:

$$\rho_{\theta-\pi}(\xi) = \rho_{\theta}(-\xi).$$

Note that:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \text{ and } \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{6}$$

The set of measurements, $\rho_\theta(\xi)$ where $\theta \in [0, \pi)$ and $\xi \in (-\infty, \infty)$ represents the view data or the sinogram in the 2-D space spanned by $\xi$ and $\theta$,

$$\rho(\xi, \theta) = \rho_\theta(\xi). \tag{7}$$
Our goal is to recover $\mu(x, y)$ from $\rho(\xi, \theta)$.

**Remark:** The Radon Transform is integration along a line. We can specify a line by its distance $\xi$ from the origin and the angle it makes with the x-axis $\theta$. A point $(x, y)$ is on the line if and only if $(x, y) \cdot (\cos \theta, \sin \theta) = \xi$. Thus in a 2D plane $\delta(x \cos \theta + y \sin \theta - \xi)$ is non-zero only when the point $(x, y)$ is on this line. Thus, we see the Radon Transform.

Alternatively, in polar notation, a point $(r, \phi)$ is on the line if and only if $(r \cos \phi, r \sin \phi)(\cos \theta, \sin \theta) = \xi$ which reduces to $r \cos(\phi - \theta) = \xi$. This gives an expression for the Radon Transform in polar notation

$$\rho(\xi, \theta) = \int_{0}^{\pi} \int_{-\infty}^{\infty} \mu(r, \phi) \delta(r \cos(\phi - \theta) - \xi) r dr d\phi$$

also written as

$$\rho(\xi, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x, y) \delta(x \cos \theta + y \sin \theta - \xi) dx dy.$$

### 1.4 Understanding Back Projection

[Diagram of back projection]

Figure 4: Back projection sums over measurements along rays passing through a fixed point as an estimate of $\mu$ there.
The idea of backprojection goes back to *Linear and Transaxial Tomography*. Consider the same idea with a single source and a single sensor rotating around the object around some fixed point $\rho(x_0, y_0)$, Figure(4). The idea is to backproject each view data to all points giving rise to it in an accumulative fashion, i.e., the estimated linear attenuation coefficient at each point in the sum of intensities measured for all rays going through the point. The sensor reading for each ray through $\rho$, indexed by $\theta$, is obtained from Equation (4)

$$
\rho(\xi_0, \theta) = \int_{-\infty}^{\infty} \mu(\xi_0 \cos \theta - \eta \sin \theta, \xi_0 \sin \theta + \eta \cos \theta) d\eta, \quad \text{(8)}
$$

where $\xi_0 = x_0 \cos \theta + y_0 \sin \theta$. Each ray passes through $\rho$, but any other point participates in the view data in only one ray. Thus, in the cumulative response, the contribution of all points fades in comparison to that of $\rho$. The process of accumulating all ray measurements through $\rho$ and using it as an estimate of $\mu(x_0, y_0)$ at $\rho$ is called *backprojection*. Formally,

$$
\mu_{BP}(x_0, y_0) = \frac{1}{\pi} \int_{0}^{\pi} \rho(\xi_0, \theta) d\theta, \quad \text{where } \xi_0 = x_0 \cos \theta + y_0 \sin \theta. \quad \text{(9)}
$$

**How good an estimate of $\mu$ is $\mu_{BP}$?** The following theorem relates the backprojection estimate of $\mu$ to $\mu$.

**Theorem 1.**

$$
\mu_{BP}(x, y) = \mu(x, y) * \frac{1}{\pi} \frac{1}{\sqrt{x^2 + y^2}}. \quad \text{(10)}
$$

**Proof.** We can relate the two by simplifying Equation (9) using Equation (8):

$$
\mu_{BP}(x_0, y_0) = \frac{1}{\pi} \int_{0}^{\pi} \left[ \int_{-\infty}^{\infty} \mu(\xi_0(\theta) \cos \theta - \eta \sin \theta, \xi_0(\theta) \sin \theta + \eta \cos \theta) d\eta \right] d\theta, \quad \text{(11)}
$$

where $\xi_0(\theta) = x_0 \cos \theta + y_0 \sin \theta$. We can gain a better insight into this integral by switching to a polar coordinate system centered around $(x_0, y_0)$,

$$
\begin{align*}
\begin{cases}
x = x_0 + r \cos \phi \\
y = y_0 + r \sin \phi.
\end{cases}
\end{align*} \quad \text{(12)}
$$

Now, given $(\theta, \eta)$ in integral (11), these identify $(x, y)$

$$
\begin{align*}
\begin{cases}
x = \xi_0(\theta) \cos \theta - \eta \sin \theta \\
y = \xi_0(\theta) \sin \theta + \eta \cos \theta.
\end{cases}
\end{align*} \quad \text{(13)}
$$
Equating (12) and (13) relates \((r, \phi)\) to \((\eta, \theta)\):

\[
\begin{align*}
\begin{cases}
  r \cos \phi &= \xi_0(\theta) \cos \theta - \eta \sin \theta - x_0 = (\xi_0(\theta) \cos \theta - \eta \sin \theta) - (\xi_0(\theta) \cos \theta - \eta_0(\theta) \sin \theta) \\
  r \sin \phi &= \xi_0(\theta) \sin \theta + \eta \cos \theta - y_0 = (\xi_0(\theta) \sin \theta + \eta \cos \theta) - (\xi_0(\theta) \sin \theta + \eta_0(\theta) \cos \theta)
\end{cases}
\end{align*}
\]

or,

\[
\begin{align*}
\begin{cases}
  r \cos \phi &= -(\eta - \eta_0(\theta)) \sin \theta \\
  r \sin \phi &= (\eta - \eta_0(\theta)) \cos \theta
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
\xi_0(\theta) &= x_0 \cos \theta + y_0 \sin \theta \\
\eta_0(\theta) &= -x_0 \sin \theta + y_0 \cos \theta.
\end{align*}
\]

When \(\eta < \eta_0(\theta)\), we have

\[
\begin{align*}
\begin{cases}
  r &= -(\eta - \eta_0(\theta)) \\
  \phi &= \theta - \frac{\pi}{2}
\end{cases}
\end{align*}
\]

or, solving for \((\eta, \theta)\)

\[
\begin{align*}
\begin{cases}
  \eta &= -r - x_0 \cos \phi - y_0 \sin \phi \\
  \theta &= \phi + \pi/2
\end{cases}
\end{align*}
\]

When \(\eta > \eta_0(\theta)\), we have

\[
\begin{align*}
\begin{cases}
  r &= \eta - \eta_0(\theta) \\
  \phi &= \theta + \frac{\pi}{2}
\end{cases}
\end{align*}
\]

or, solving for \((\eta, \theta)\)

\[
\begin{align*}
\begin{cases}
  \eta &= r + x_0 \cos \phi + y_0 \sin \phi \\
  \theta &= \phi - \pi/2
\end{cases}
\end{align*}
\]

In rewriting (9) in terms of \(r\) and \(\phi\), we will also need an expression for \(\xi_0(\theta)\).

When \(\eta < \eta_0(\theta)\)

\[
\xi_0(\theta) = x_0 \cos(\phi + \frac{\pi}{2}) + y_0 \sin(\phi + \frac{\pi}{2})
\]

\[= -x_0 \sin \phi + y_0 \cos \phi.
\]

10
We can now rewrite \( \mu_{BP} \) in (9) as

\[
\mu_{BP}(x_0, y_0) = \frac{1}{\pi} \int_0^\pi \int_{-\infty}^{\eta_0(\theta)} \mu(x_0 \cos \phi - y_0 \sin \phi, \text{ } x_0 \sin \phi + y_0 \cos \phi) \cos \theta \text{ } d\eta d\theta
\]

\[
+ \frac{1}{\pi} \int_0^\pi \int_{-\infty}^{\eta_0(\theta)} \mu(x_0 \cos \phi - y_0 \sin \phi, \text{ } x_0 \sin \phi + y_0 \cos \phi) \sin \theta \text{ } d\eta d\theta
\]

\[
= \frac{1}{\pi} \int_0^\pi \int_{-\infty}^{\eta_0(\theta)} \mu(x_0 \cos \phi - y_0 \sin \phi, \text{ } x_0 \sin \phi + y_0 \cos \phi) \cos \theta \sin \phi + (x_0 \cos \phi + y_0 \sin \phi) \cos \phi \text{ } d\eta d\theta
\]

\[
+ \frac{1}{\pi} \int_0^\pi \int_{-\infty}^{\eta_0(\theta)} \mu(x_0 \cos \phi - y_0 \sin \phi, \text{ } x_0 \sin \phi + y_0 \cos \phi) \sin \phi + (x_0 \cos \phi + y_0 \sin \phi) \sin \phi + r \text{ } d\eta d\theta
\]

\[
= \frac{1}{\pi} \int_0^\pi \int_{-\infty}^{\eta_0(\theta)} \mu(x_0 + r \cos \phi, \text{ } y_0 + r \sin \phi) \text{ } dr d\phi
\]

We are now in a position to interpret this as an area integral, i.e.

\[
\mu_{BP}(x_0, y_0) = \frac{1}{\pi} \int_0^{2\pi} \int_0^\infty \left[ \frac{\mu(x_0 + r \cos \phi, \text{ } y_0 + r \sin \phi)}{r} \right] \text{ } dr d\phi
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mu(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \text{ } dx dy
\]

\[
= \mu(x, y) \ast \frac{1}{\pi} \left. \frac{1}{\sqrt{x^2 + y^2}} \right|_{x=x_0, y=y_0},
\]

which is the \( \frac{1}{r} \) filter well-known from the Fourier approach to the CT reconstruction. In summary, the back projection process gives a blurred estimate of \( \mu(x, y) \) with a \( \frac{1}{r} \) blurring filter.
1.5 The Fourier Approach to CT Reconstruction

In this approach, the CT-reconstruction of \( \mu(x, y) \) relies on its Fourier Transform,

\[
\mathcal{F}\{\mu(x, y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x, y) e^{-i\omega x} e^{-i\omega y} dx dy.
\]

Consider a polar representation of \( \mu \) in the Fourier domain, i.e., \( \mathcal{F}\{\mu(x, y)\} \), where \( \omega_x = \omega \cos \theta \) and \( \omega_y = \omega \sin \theta \). We have

\[
\mathcal{F}\{\mu(x, y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x, y) e^{-i\omega x} e^{-i\omega y} dx dy
\]

The latter integral can be viewed in the form of a 1-D Fourier Transform where \( \theta \) is fixed, but \( \omega \) is varying:

\[
\mathcal{F}\{\rho(\xi, \theta)\}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\xi, \theta)e^{-i\omega \xi} d\xi
\]

where \( \rho(\xi, \theta) \) is the one dimensional Fourier Transform of \( \rho(\omega, \theta) \) for fixed \( \theta \) along the variable \( \xi \). In other words, when keeping \( \theta \) fixed in the Fourier domain, this slice of the Fourier Transform of \( \mathcal{F}\{\mu(x, y)\} \) corresponds exactly to the 1-D Fourier Transform of the Radon Transform for that \( \theta \), i.e., \( \rho(\xi, \theta) \).

1.6 Filtered Backprojection

The central slice theorem allows us to recover \( \mu(x, y) \) from the measurements using the inverse Fourier Transform and a change of coordinates from the
Cartesian coordinates \((\omega_x, \omega_y)\) to the polar coordinates \(\omega, \theta\).

\[
\mu(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{\mu}(\omega_x, \omega_y) e^{i\omega x} e^{i\omega y} d\omega_x d\omega_y
\]
\[
= \frac{1}{4\pi^2} \int_0^{\pi} \int_{-\infty}^{\infty} \bar{\mu}(\omega, \theta) e^{i\omega (\cos \theta x + \sin \theta y)} |\omega| d\omega d\theta
\]
\[
= \frac{1}{2\pi} \int_0^{\pi} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\mu}(\omega, \theta) |\omega| e^{i\omega (x \cos \theta + y \sin \theta)} d\omega \right] d\theta. \quad (14)
\]

The inner integral in the square brackets is the one-dimensional inverse Fourier Transform of \(\bar{\mu}(\omega, \theta)|\omega|\) along the variable \(\omega\) with fixed \(\theta\). Define

\[
\rho^*(\xi, \theta) = \mathcal{F}^{-1}[\bar{\mu}(\omega, \theta)|\omega|] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\mu}(\omega, \theta) |\omega| e^{i\omega (x \cos \theta + y \sin \theta)} d\omega. \quad (15)
\]

Then

\[
\mu(x, y) = \frac{1}{2\pi} \int_0^{\pi} \rho^*_\theta(x \cos \theta + y \sin \theta) d\theta. \quad (16)
\]

Equation (16) provided the recipe for recovery of \(\mu\):

1) **Filtering step:** From \(\rho(\xi, \theta)\), find \(\rho^*(\xi, \theta) = \mathcal{F}^{-1}_\omega \{\mathcal{F}_\xi \{\rho(\xi, \theta)\}\}|\omega|\},\)

2) **Backprojection:** Sum over \(\rho^*(\xi, \theta)\) over all \(\theta\).

The first step is called the filtering step because it seems to be interpretable as a convolution: Observing that a product in the Fourier domain is a convolution in the spatial domain it is tempting to write

\[
\rho^*(\xi, \theta) = \rho(\xi, \theta) * \mathcal{F}^{-1}_\omega \{|\omega|\}. \quad (17)
\]

However, the function \(|\omega|\) has infinite energy and such a split into two functions is not valid. A different split is possible:

\[
\mathcal{F}^{-1}_\omega \{\mathcal{F}_\xi \{\rho(\xi, \theta)\}\}|\omega|\} = \mathcal{F}^{-1}_\omega \{\mathcal{F}_\xi \{\rho(\xi, \theta)\}\ \frac{H(w)}{|w|}\} \{H(w)|\omega|\}\}
\]
\[
= \mathcal{F}^{-1}_\omega \{\mathcal{F}_\xi \{\rho(\xi, \theta)\}\ \frac{H(w)}{H(w)}\} * \mathcal{F}^{-1}_\omega \{H(w)|\omega|\}, \quad (18)
\]

where \(H(\omega)\) is expected to decrease sufficiently fast to compensate for the increase by \(|\omega|\) so that \(\mathcal{F}^{-1}_\omega \{H(w)|\omega|\}\) exists. If \(\rho(\xi, \theta)\) is band-limited so that frequencies beyond \(\Omega\) are not present,

\[
\mathcal{F}_\xi \{\rho(\xi, \theta)\} = 0 \text{ when } \omega > \Omega, \quad (20)
\]

then we can design \(H(\omega)\) to be 1 up to \(\Omega\), and then drop monotonically to zero.
\[
\begin{cases}
H(\omega) = 1, & |\omega| \leq \Omega \\
H(\omega) \neq 0, & \text{otherwise}.
\end{cases}
\] (21)

In this case we have we have
\[
\mathcal{F}_\omega^{-1}\left\{\frac{\mathcal{F}_\xi\{\rho(\xi, \theta)\}}{H(w)}\right\} = \mathcal{F}_\omega^{-1}\left\{\frac{\mathcal{F}_\xi\{\rho(\xi, \theta)\}}{1}\right\} = \rho(\xi, \theta),
\] (22)

and
\[
\rho^*(\xi, \theta) = \rho(\xi, \theta) \ast \mathcal{F}_\omega^{-1}\{H(w)|\omega|\}. \quad (23)
\]

The filter \(H(\omega)\) can be designed a number of ways. First, the filter defined by Ramachandran and Lakshminarayanan [23], known now as the Ram-Lak filter, is simply obtained by limiting the frequency domain response, Figure(5).

\[
H(\omega) = \begin{cases} 
1, & |\omega| \leq \Omega \\
0, & \text{otherwise}.
\end{cases}
\] (24)

so that the overall filter is \(H_{RL}(\omega) = H_{RL}(\omega)|\omega|\), or,

\[
H_{RL}(\omega) = \begin{cases} 
|\omega|, & |\omega| \leq \Omega \\
0, & \text{otherwise}.
\end{cases}
\] (25)

Using Appendix(C), Formula 3, the spatial domain filter is

\[
h_{RL}(x) = \frac{\Omega^2}{\pi}[\text{sinc}(\Omega x) - \frac{1}{2}\text{sinc}^2\left(\frac{\Omega x}{2}\right)],
\] (26)

See Figure(5).

Second, the Shepp-Logan filter [25] has a similar design that it also limits the frequency upper bound, but in contrast to the Ram-Lak filter, it does so gradually using a \(\text{sinc}\) function \(\frac{2\pi}{\Omega} \text{sinc}\left(\frac{\pi}{2}\right),\) Figure(5), with the overall filter defined as

\[
H_{SL}(\omega) = \begin{cases} 
|\sin\left(\frac{\pi\omega}{2\Omega}\right)|, & |\omega| < \Omega \\
0, & \text{otherwise}
\end{cases}
\] (27)
Figure 5: (a) Examples of the band-limited filter function of sampled data. Note the cyclic repetitiveness of the digital filter. (b) Spatial domain filter kernels corresponding to the filter functions shown in the Ram-Lak filter is a high-pass filter with a sharp response but results in some noise enhancement, while the Shepp-Logan and the Hamming window filters are noise-smoothed filters and therefore have better SNR. (Images taken from [3])

The spatial filter corresponding to the Shepp-Logan filter is

$$h_{SL}(\omega) = \frac{1}{2\pi} \int_{-\omega}^{\omega} H_{SL}(\omega) e^{i\omega x} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega}^{\omega} \left| \sin \left( \frac{\pi \omega}{2\Omega} \right) \right| e^{i\omega x} d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{\omega} 2 \sin \left( \frac{\pi \omega}{2\Omega} \right) \cos \omega x d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{\omega} \sin \left( \left( \frac{\pi}{2\Omega} + x \right) \omega \right) d\omega + \frac{1}{2\pi} \int_{0}^{\omega} \sin \left( \left( \frac{\pi}{2\Omega} - x \right) \omega \right) d\omega$$

$$= \frac{-1}{2\pi} \cos \left( \left( \frac{\pi}{2\Omega} + x \right) \omega \right) \bigg|_{0}^{\omega} - \frac{1}{2\pi} \cos \left( \left( \frac{\pi}{2\Omega} - x \right) \omega \right) \bigg|_{0}^{\omega}$$

$$= \frac{1}{2\pi} \left[ \frac{1}{\frac{\pi}{2\Omega} + x} + \frac{1}{\frac{\pi}{2\Omega} - x} \right]$$

$$= \frac{1}{2\pi} \left[ \frac{\sin(\Omega x)}{\frac{\pi}{2\Omega} + x} + \frac{\sin(\Omega x)}{\frac{\pi}{2\Omega} - x} \right]$$

$$= \frac{1}{2\pi} \left[ \frac{\frac{\pi}{2\Omega} - x}{(\frac{\pi}{2\Omega})^2 - x^2} \right] \left[ \frac{\pi}{\Omega} - 2x \sin(\Omega x) \right]$$

$$= \frac{1}{\pi} \left[ \frac{\frac{\pi}{2\Omega} - x \sin(\Omega x)}{(\frac{\pi}{2\Omega})^2 - x^2} \right].$$
The Ram-Lak filter has the following formula in the spatial domain when sampled at Nyquist rate which is two times the bandwidth of a bandlimited signal so that \( \Delta x = \frac{\pi}{\Omega} \):

\[
H(k) = \begin{cases} 
\frac{\pi}{4}, & \text{if } k=0 \\
\frac{1}{\pi^2 k^2}, & \text{if } k \text{ odd} \\
0, & \text{if even}
\end{cases}
\]

while the Shepp-Logan filter is given by:

\[
H(k) = \frac{-8\Omega^2}{\pi^2(4k^2 - 1)}, \quad k = 0, \pm 1, \pm 2, \ldots
\]

in the spatial domain.

Explain the Hamming Window approach proposed originally by Richard Hamming, using the raised cosine:

\[
w[n] = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right).
\]

\[\text{(28)}\]

In a different approach, Rosenfeld and Kak [24] use \( H(\omega) = e^{-\epsilon|\omega|} \) so that the overall filter is \( H_\epsilon(\omega) = |\omega|e^{-\epsilon|\omega|} \). In the spatial domain this filter is written as \( h_\epsilon(x) = \frac{e^{2-(2\pi x)^2}}{(2\pi x)^2} \). This filter is plotted in Figure(6) for several values of \( \epsilon \).

It is not clear why a filter that balances being 1 in the range up to Omega and smaller after that has not been used:

\[
H_{Kim} = \frac{\omega}{1 + (\frac{\omega}{\Omega})^6}.
\]

\[\text{(29)}\]

Note: Add a formal definition of backprojection as “Adjoint Radon Transform.”
1.7 Watch out for those singularities: A review of Calculus

What do you do when the integrand has a singularity? Consider the integral

\[ I = \int_{-a}^{a} \frac{1}{\sqrt{x}} \, dx, \]  

(30)

where the integrand blows up at \( x = 0 \); what happens to the value of the integral? A substitution \( y = \sqrt{x} \) gives

\[ I = \frac{1}{2} \int_{-\sqrt{a}}^{\sqrt{a}} \, dy = \sqrt{a}, \]  

(31)

a finite value, even when \( a \to 0 \). Now consider the integral

\[ II = \int_{-a}^{a} \frac{1}{x^2} \, dx. \]  

(32)
This can be integrated directly, giving

\[ II = - \frac{1}{x} \bigg|_{-a}^{a} = - \frac{2}{a}, \]  

which goes to infinity as \( a \to 0 \). In such cases, the evaluation above does not hold because of the singularity. Instead, we must use the Cauchy Principal Value which states that

\[ PV \int_{a}^{b} f(x) \, dx = \lim_{\epsilon \to \infty} \left[ \int_{a}^{c-\epsilon} f(x) \, dx + \int_{c+\epsilon}^{b} f(x) \, dx \right]. \]  

(34)

The **Hilbert Transform** pair are defined as

\[ g(y) = PV \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{x-y} \, dx, \]  

(35)

\[ f(x) = PV \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(y)}{y-x} \, dy. \]  

(36)

### 1.8 The Spatial Domain Approach to CT Reconstruction

Radon\cite{22} showed that

\[ \mu(x, y) = \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{-\infty}^{\infty} \frac{\partial \rho}{\partial \xi} x \cos \theta + y \sin \theta - \xi \, d\xi \, d\theta \]  

(37)

The following is a summary of his elegant argument. [to summarize]

Let \( \xi = x \cos \theta + y \sin \theta \) be the \( \xi \)-coordinate of \( \mu(x, y) \). Then, \( x \cos \theta + y \sin \theta - \xi = \xi - \xi \) indicates the distance of the point \( M(x, y) \) from the X-ray beam identified by \( (\xi, \theta) \). Observe that one can view the inner integral as the convolution of \( \rho_\xi \) with the filter \( \frac{1}{\xi} \).

\[
\tilde{\rho}(\xi) = (\rho_\xi * \frac{1}{\xi})(\xi) = \int_{-\infty}^{\infty} \rho_\xi(\xi) \frac{1}{\xi - \xi} \, d\xi,
\]

(38)

so that

\[ \mu(x, y) = \frac{1}{4\pi^2} \int_{0}^{2\pi} \tilde{\rho}(\xi) \, d\theta. \]
When the points are represented in polar representation \( M(r, \phi) \), i.e. \((x, y) = (r \cos \phi, r \sin \phi)\), we have

\[
x \cos \theta + y \sin \theta - \xi = r \cos \phi \cos \theta + r \sin \phi \sin \theta = r \cos(\theta - \phi) - \xi.
\]

Thus, the Radon Inversion formula in polar form is

\[
\mu(r, \phi) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{\partial \rho}{\partial \xi}(\xi, \theta) \frac{r \cos(\theta - \phi) - \xi}{\xi} d\xi d\theta. \tag{39}
\]

Alternatively, one can reparametrize \( \hat{\xi} = r \cos(\theta - \phi) - \xi \) in the inner integral to get

\[
\mu(r, \phi) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{\partial \rho}{\partial \xi}(r \cos(\theta - \phi) + \hat{\xi}, \theta) \frac{\hat{\xi}}{\xi} d\hat{\xi} d\theta. \tag{40}
\]

Unfortunately, the denominator for each \( \theta \) will vanish at \( \hat{\xi} = 0 \) for Equation (40) and \( \xi = x \cos \theta + y \sin \theta \) for Equation (37). Ben to expand on this

**Remark:** As Horn [13] noted the Radon’s inversion formula containing \( \rho_\xi \) does not strictly apply when \( \rho_\xi \) is not continuous. Instead, one rewrites the inner integral by integration by parts as:

\[
\int_{-\infty}^{\infty} \frac{\rho_\xi}{\xi - \xi} d\xi = \lim_{\epsilon \to 0} \left[ \int_{-\infty}^{\xi - \epsilon} \frac{\rho_\xi}{\xi - \xi} d\xi + \int_{\xi + \epsilon}^{\infty} \frac{\rho_\xi}{\xi - \xi} d\xi \right]
\]

\[
= \lim_{\epsilon \to 0} \left[ \rho \left( \frac{\xi - \epsilon}{\xi - \xi} - \frac{\xi + \epsilon}{\xi - \xi} \right) - \int_{-\infty}^{\xi - \epsilon} \frac{\rho}{(\xi - \xi)^2} d\xi + \int_{\xi + \epsilon}^{\infty} \frac{\rho}{(\xi - \xi)^2} d\xi \right] = \int_{-\infty}^{\infty} \frac{\rho(\xi, \theta)}{\xi - \xi} d\xi
\]

\[
= \int_{-\infty}^{\infty} \frac{\rho(\xi, \theta)}{\xi - \xi} d\xi = \rho * \xi F_\epsilon(\xi - \xi) d\xi
\]

\[
= \rho * \xi F_\epsilon, \tag{41}
\]
where
\[ F_\epsilon(\xi) = \begin{cases} \frac{-1}{\xi^2} & |\xi| \geq \epsilon \\ \frac{1}{\epsilon^2} & |\xi| < \epsilon, \end{cases} \] (42)
as shown in Figure(7).

Figure 7: Plot of \( F_\epsilon(\xi) \) for \( \epsilon = (0.5, 0.2, 0.1) \).

Then,
\[
\mu(x, y) = \lim_{\epsilon \to 0} \frac{1}{2\pi^2} \int_{0}^{\pi} \int_{-\infty}^{\infty} \rho(\xi, \theta) F_\epsilon(x \cos \theta + y \sin \theta - \xi) d\theta \\
= \lim_{\epsilon \to 0} \frac{1}{2\pi^2} \int_{0}^{\pi} \left[ \rho(\xi, \theta) *_{\xi} F_\epsilon \right](x \cos \theta + y \sin \theta - \xi) d\theta \quad (43)
\]

Figure(8) shows a reconstruction using this method.

1.9 Inverse Radon Transform

Ben to write this using a summary of Radon’s 1917 paper and Horn’s papers
2 Fan Beam Geometry

In the fan beam geometry, all rays emanate from a single source $S$ located at polar coordinates $(D_0, \beta + \pi/2)$, where $D_0 = |OS|$ and $\beta$ is the angle of a virtual detector array passing through $O$. The actual detector array is some distance $D_1$ away from origin, but the analysis is simplified if the measurements are scaled down to pass through the origin $O$, Figure(9). An arbitrary point $M$ is described in polar coordinates $(r, \phi)$. The source detector assembly $S$ is assumed to rotate around the origin so that $\beta \in [0, 2\pi]$. Thus, the sinogram collected in this geometry is a 2D set of line integral measurements: for each $\beta$ the line integrals for all $A$, as specified by $\xi$, are reported. Let $g(\xi, \beta)$ denote the line integrals. Note that $|\xi| < D_0$. The sinogram specified by $g(\xi, \beta)$ is natural when the samples are along a line perpendicular to $OS$. However, when detectors are arranged in a circular arc centered at $S$, the angle $\alpha = \angle OSA$ is a better description of the detectors, and the sinogram
Figure 9: The fan beam geometry is illustrated for a point source $S$ and a virtual linear detection array aligned with the $\xi$-axis. The actual detector array is a constant distance array.

in the form of $\hat{g}(\alpha, \beta)$ is used. The two representations are interchangeable:

$$\begin{aligned} \hat{g}(\alpha, \beta) &= g(D_0 \tan \alpha, \beta) \\ g(\xi, \beta) &= \hat{g}\left(\tan^{-1}\left(\frac{\xi}{D_0}\right), \beta\right). \end{aligned}$$ \ (44)

**Proposition 1** (Rebinning from fan beam geometry to parallel beam geometry).

$$\rho(\hat{\xi}, \theta) = g(\xi, \beta),$$ \ (45)

where

$$\begin{aligned} \xi &= \frac{D_0 \hat{\xi}}{\sqrt{D_0^2 - \hat{\xi}^2}} \quad \text{and} \quad \hat{\xi} = \frac{D_0 \xi}{\sqrt{D_0^2 + \xi^2}} \\ \beta &= \theta - \sin^{-1}\left(\frac{\hat{\xi}}{D_0}\right) \quad \text{and} \quad \theta = \beta + \tan^{-1}\left(\frac{\xi}{D_0}\right). \end{aligned}$$ \ (46)
Proof. We have three relationships from the geometry in Figure(10)

\[
\begin{align*}
\cos \alpha &= \frac{\hat{\xi}}{\xi} \\
\sin \alpha &= \frac{\hat{\xi}}{D_0} \\
\theta &= \beta + \alpha.
\end{align*}
\]

By squaring the first two equations we have

\[
\frac{\hat{\xi}^2}{\xi^2} + \frac{\hat{\xi}^2}{D_0^2} = 1,
\]

which gives $\xi$ in terms of $\hat{\xi}$ and vice versa. Observing that $\tan \alpha = \frac{\hat{\xi}}{D_0}$ gives the remaining equations. \qed
Corollary 1. We have

\[
\begin{align*}
\alpha(\hat{\xi}, \theta) &= \sin^{-1}\left( \frac{\hat{\xi}}{D_0} \right) \\
\beta(\hat{\xi}, \theta) &= \theta - \sin^{-1}\left( \frac{\hat{\xi}}{D_0} \right)
\end{align*}
\]

and

\[
\begin{align*}
\hat{\xi}(\alpha, \beta) &= D_0 \sin \alpha \\
\theta(\alpha, \beta) &= \beta + \alpha.
\end{align*}
\]

Thus,

\[
\begin{align*}
\rho(\hat{\xi}, \theta) &= \check{g}\left( \sin^{-1}\left( \frac{\hat{\xi}}{D_0} \right), \theta - \sin^{-1}\left( \frac{\hat{\xi}}{D_0} \right) \right) \\
\hat{g}(\alpha, \beta) &= \rho(D_0 \sin \alpha, \beta + \alpha).
\end{align*}
\]

Proof. We have from Proposition(45) that

\[
\begin{align*}
\rho(\hat{\xi}, \theta) &= g(\xi, \beta) \\
&= \check{g}\left( \tan^{-1}\left( \frac{\xi}{D_0} \right), \beta \right) \\
&= \check{g}\left( \tan^{-1}\left( \frac{\hat{\xi}}{\sqrt{D_0^2 + \hat{\xi}^2}} \right), \theta - \sin^{-1}\left( \frac{\hat{\xi}}{D_0} \right) \right) \\
&= \check{g}\left( \sin^{-1}\left( \frac{\hat{\xi}}{D_0} \right), \theta - \sin^{-1}\left( \frac{\hat{\xi}}{D_0} \right) \right).
\end{align*}
\]

Conversely, by Proposition(1)

\[
\begin{align*}
\hat{g}(\alpha, \beta) &= g(D_0 \tan \alpha, \beta) \\
&= \rho\left( \frac{D_0(D_0 \tan \alpha)}{\sqrt{D_0^2 + (D_0 \tan \alpha)^2}}, \beta + \alpha \right) \\
&= \rho(D_0 \sin \alpha, \beta + \alpha).
\end{align*}
\]

\[
\square
\]

2.1 Radon Transform and Fan Beam Geometry

Deriving an expression to describe fan beam measurements requires a careful examination. An X-ray measurement in the parallel beam geometry,
Figure (11a) indicates the attenuation of X-ray intensity through a narrow beam with $\Delta \xi$, \textit{i.e.}

$$
\int_{\xi_0 - \frac{\Delta \xi}{2}}^{\xi_0 + \frac{\Delta \xi}{2}} \int_{-\infty}^{\infty} \mu(x(\xi, \eta), y(\xi, \eta)) d\eta d\xi
$$

which in the limit as $\Delta \xi \to 0$ is asymptotically

$$
\Delta \xi \int_{-\infty}^{\infty} \mu(x(\xi_0, \eta), y(\xi_0, \eta)) d\eta,
$$

(47)

with the latter part corresponding to the idealized Radon Transform. However, an X-ray measurement in the fan beam geometry corresponds fans out so one might infer the measurement is

$$
\int_{\alpha_0 - \frac{\Delta \alpha}{2}}^{\alpha_0 + \frac{\Delta \alpha}{2}} \int_{0}^{\infty} \mu(x(\tilde{\eta}, \alpha), y(\tilde{\eta}, \alpha)) \tilde{\eta} d\tilde{\eta} d\alpha,
$$

where $\tilde{\eta}$ is the distance from $S$ to the point $P$ along the ray $SA$. Rather, the beam density also decreases along the front by the same $\tilde{\eta}$ factor so that the expression is asymptotically equivalent to

$$
\Delta \alpha \int_{0}^{\infty} \mu(x(\tilde{\eta}, \alpha), y(\tilde{\eta}, \alpha)) d\tilde{\eta},
$$

(48)

giving again the Radon line integral. What this analysis shows, however, is that numerically the fan beam focuses on smaller regions initially and on larger regions as the ray propagates. When the image is not varying rapidly, this does not present a problem. However, when the variation approaches voxel size, numerical approximation may not be faithful to the true values.

The expression of the Radon Transform in the fan beam geometry can now be easily derived by noting a point along the line $SA$ is specified by

$$
P = \xi(\cos \beta, \sin \beta) + \tilde{\eta}(\cos(\beta + \alpha + \pi/2), \sin(\beta + \alpha + \pi/2))
= (\xi \cos \beta - \tilde{\eta} \sin(\beta + \alpha), \xi \sin \beta + \tilde{\eta} \cos(\beta + \alpha)),
$$

where $\tilde{\eta}$ is length parameter along $SA$. It is easy to see that $\tilde{\eta} = \frac{\eta}{\cos \alpha}$. Thus,

$$
g(\xi, \beta) = \int_{-\infty}^{\infty} \mu(\xi \cos \beta - \frac{\eta}{\cos \alpha} \sin(\beta + \alpha), \xi \sin \beta + \frac{\eta}{\cos \alpha} \cos(\beta + \alpha)) \frac{d\eta}{\cos \alpha}
= \frac{1}{\cos \alpha} \int_{-\infty}^{\infty} \mu(\xi \cos \beta - \eta \sin \beta - \eta \cos \beta \tan \alpha, \xi \sin \alpha + \eta \cos \beta - \eta \sin \beta \tan \alpha) d\eta.
$$
Similarly,

\[ \hat{g}(\alpha, \beta) = g(D_0 \tan \alpha, \beta) = \frac{1}{\cos \alpha} \int_{-\infty}^{\infty} \mu((D_0 - \eta) \cos \beta \tan \alpha - \eta \sin \beta, (D_0 - \eta) \sin \beta \tan \alpha + \eta \cos \beta) d\eta. \]

Figure 11: The relationship between x-ray attenuation measurements and the Radon Transform. (a) Parallel beam (b) Fan beam.

2.2 Radon Inversion Formula for Fan Beam Geometry

Proposition 2.

\[
\mu(r, \phi) = \frac{1}{4\pi^2} \int_0^{2\pi} \frac{1}{D} \left[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\hat{g}_\alpha - \hat{g}_\beta}{\sin(\overline{\alpha} - \alpha)} d\alpha \right] d\beta, \quad (49)
\]

where \( D = \sqrt{D_0^2 + r^2 + 2rD_0 \sin(\beta - \phi)} \) is the distance from \( M(r, \phi) \) to the source \( S(D_0, \beta + \frac{\pi}{2}) \) and \( \overline{\alpha} = \tan^{-1} \left[ \frac{r \cos(\beta - \phi)}{D_0 + r \sin(\beta - \phi)} \right] \) is the angle \( \angle OSM \).

Proof. Let \( M(r, \phi) \) be an arbitrary point, Figure(9). Let \( D \) denote its distance to source \( D = |SM| \), so that we have via the triangle inequality in the triangle \( SOM \)

\[
D^2 = D_0^2 + r^2 - 2rD_0 \cos \left( \beta + \frac{\pi}{2} - \phi \right) = D_0^2 + r^2 + 2rD_0 \sin(\beta - \phi).
\]
Also, let $\alpha$ be the angle of the ray through $M$, i.e., $\alpha = \angle(SO, SM)$. Then,

$$\tan(\alpha) = \frac{MF}{SF} = \frac{r \sin \left( \beta + \frac{\pi}{2} - \phi \right)}{D_0 - r \cos \left( \beta + \frac{\pi}{2} - \phi \right)} = \frac{r \cos(\beta - \phi)}{D_0 + r \sin(\beta - \phi)}. \quad (50)$$

Note that this gives the polar representation of $M$ in the coordinates of source $S$ with respect to the direction $SO$.

Radon’s inversion formula in polar notation was derived in Equation (39). This can be represented in the fan beam geometry as follows. First, we can obtain an expression for $\rho(\hat{\xi}, \theta)$ using Corollary (1)

$$\rho(\hat{\xi}, \theta) = \hat{g}(\alpha(\hat{\xi}, \theta), \beta(\hat{\xi}, \theta)),$$

where $\alpha(\hat{\xi}, \theta) = \sin^{-1}(\frac{\hat{\xi}}{D_0})$ and $\beta(\hat{\xi}, \theta) = \theta - \sin^{-1}(\frac{\hat{\xi}}{D_0})$. Then

$$\frac{\partial \rho}{\partial \xi}(\hat{\xi}, \theta) = \frac{\partial \hat{g}}{\partial \alpha}(\alpha(\hat{\xi}, \theta), \beta(\hat{\xi}, \theta)) \frac{\partial \alpha}{\partial \xi}(\hat{\xi}, \theta) + \frac{\partial \hat{g}}{\partial \beta}(\alpha(\hat{\xi}, \theta), \beta(\hat{\xi}, \theta)) \frac{\partial \beta}{\partial \xi}(\hat{\xi}, \theta). \quad (51)$$

Since $\sin \alpha(\hat{\xi}, \theta) = \frac{\hat{\xi}}{D_0}$, we can differentiate to get $\cos \alpha \hat{\xi} = \frac{1}{D_0}$ leading to

$$\left\{ \begin{array}{l} \frac{\partial \alpha}{\partial \xi}(\hat{\xi}, \theta) = \frac{1}{D_0 \cos \alpha(\hat{\xi}, \theta)} \\ \frac{\partial \beta}{\partial \xi}(\hat{\xi}, \theta) = -\frac{1}{D_0 \cos \alpha(\hat{\xi}, \theta)} \end{array} \right.$$ 

Thus,

$$\frac{\partial \beta}{\partial \xi}(\hat{\xi}, \theta) = \frac{1}{D_0 \cos \alpha} [\hat{g}_\alpha(\alpha, \beta) - \hat{g}_\beta(\alpha, \beta)].$$

Second, the coordinates $(\hat{\xi}, \theta)$ have to be changed to $(\alpha, \beta)$. The Jacobian requires $\left( \frac{\partial \alpha}{\partial \hat{\xi}}, \frac{\partial \beta}{\partial \hat{\xi}} \right)$, which we have already derived and $\left( \frac{\partial \alpha}{\partial \theta}, \frac{\partial \beta}{\partial \theta} \right)$ which are

$$\left\{ \begin{array}{l} \frac{\partial \alpha}{\partial \theta} = 0 \\ \frac{\partial \beta}{\partial \theta} = 1 \end{array} \right.$$ 

Thus

$$J = \frac{\partial (\alpha, \beta)}{\partial (\hat{\xi}, \theta)} = \left| \begin{array}{cc} \frac{1}{D_0 \cos \alpha} & 0 \\ \frac{\partial \alpha}{\partial \theta} & 1 \end{array} \right| = \frac{1}{D_0 \cos \alpha}. \quad 27$$
Thus,

\[
\mu(r, \phi) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{\rho_{\xi}}{r \cos(\theta - \phi) - \xi} d\xi d\theta
\]

\[
= \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{D_0 \cos \alpha} (\hat{g}_\alpha - \hat{g}_\beta) \left( r \cos(\beta + \alpha - \phi) - D_0 \sin \alpha \right) d\alpha d\beta
\]

\[
= \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\hat{g}_\alpha - \hat{g}_\beta}{r \cos(\beta + \alpha - \phi) - D_0 \sin \alpha} d\alpha d\beta.
\]

Observe from Figure(9) that

\[
\begin{align*}
    r \cos(\beta + \alpha - \phi) - D_0 \sin \alpha &= r \cos \alpha \cos(\phi - \beta) + r \sin \alpha \sin(\phi - \beta) - D_0 \sin \alpha \\
    &= FM \cos \alpha + OF \sin \alpha - D_0 \sin \alpha \\
    &= D \sin \alpha \cos \alpha + (D_0 - D \cos \alpha) \sin \alpha - D_0 \sin \alpha \\
    &= D(\sin \alpha \cos \alpha - \cos \alpha \sin \alpha) \\
    &= D \sin(\alpha - \alpha).
\end{align*}
\]

Thus

\[
\mu(r, \phi) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\hat{g}_\alpha - \hat{g}_\beta}{D \sin(\alpha - \alpha)} d\alpha d\beta
\]

\[
= \frac{1}{4\pi^2} \int_0^{2\pi} \frac{1}{D} \left[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\hat{g}_\alpha - \hat{g}_\beta}{\sin(\alpha - \alpha)} d\alpha \right] d\beta.
\]

\[\square\]

**Corollary 2.**

\[
\mu(r, \phi) = \frac{1}{4\pi^2} \int_0^{2\pi} \frac{\hat{G}(\alpha, \beta)}{\sqrt{D_0^2 + r^2 + 2rD_0 \sin(\beta - \phi)}} d\beta,
\]

where

\[
\hat{G}(\alpha, \beta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\hat{g}_\alpha - \hat{g}_\beta}{\sin(\alpha - \alpha)} d\alpha = (\hat{g}_\alpha - \hat{g}_\beta) *_{\alpha} \frac{1}{\sin \alpha}
\]

and \(\bar{\alpha}(\beta) = \tan^{-1} \left[ \frac{r \cos(\beta - \phi)}{D_0 + r \sin(\beta - \phi)} \right].\)

This corollary gives a reconstruction procedure from the sinogram \(\hat{g}(\alpha, \beta)\) for a point \(M(r, \phi)\): Take derivatives in the \(\alpha\) and \(\beta\) directions, subtract, and convolve with \(\frac{1}{\sin \alpha}\) for each column \(\beta = \text{constant}\), separately, to get \(\hat{G}(\beta)\). Then, \(\mu\) can be reconstructed from Formula\(52\). Figure\(2.2\) illustrates this procedure.
Can: 1. Form fan beam Radon Transform 2. Reparametrize in terms of $\alpha, \beta$ instead of $\xi, \beta$ 3. Follow the procedure in Corollary(2) 4. Compare to rebinning and reconstruction from $\rho(\xi, \theta)$.

Corollary 3.

$$
\mu(r, \phi) = \frac{1}{4\pi^2} \int_{0}^{2\pi} \frac{G(\xi, \beta)}{\sqrt{D_0^2 + r^2 + 2rD_0 \sin(\beta - \phi)}} d\beta,
$$

where

$$
G(\xi, \beta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_0 \left(1 + \frac{\xi^2}{D_0^2}\right) \frac{(g_\xi - g_\beta) \sqrt{\xi^2 + D_0^2}}{\sqrt{\xi^2 + D_0^2}} d\xi,
$$

and $\tilde{\xi} = D_0 \tan \bar{\alpha} = D_0 \left[\frac{r \cos(\beta - \phi)}{D_0 + r \sin(\beta - \phi)}\right]$.

Proof. Let $\xi = D_0 \tan \alpha$. Then

$$
\sin(\bar{\alpha} - \alpha) = \sin \bar{\alpha} \cos \alpha - \cos \bar{\alpha} \sin \alpha
$$

$$
= \frac{\xi}{\sqrt{\xi^2 + D_0^2}} \frac{D_0}{\sqrt{\xi^2 + D_0^2}} - \frac{D_0}{\sqrt{\xi^2 + D_0^2}} \frac{\xi}{\sqrt{\xi^2 + D_0^2}}
$$

$$
= D_0 \frac{\bar{\xi} - \xi}{\sqrt{\bar{\xi}^2 + D_0^2} \sqrt{\xi^2 + D_0^2}}.
$$

Also, note that

$$
\hat{g}_\alpha - \hat{g}_\beta = g_\xi D_0 (1 + \tan^2 \alpha) - g_\beta
$$

$$
= g_\xi D_0 \left(1 + \frac{\xi^2}{D_0^2}\right) - g_\beta.
$$

Then, using the substitution $\xi = D_0 \tan \alpha$ and $\bar{\xi} = D_0 \tan \bar{\alpha}$ in the integral of Equation(53) defines

$$
G(\xi, \beta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_0 \left(1 + \frac{\xi^2}{D_0^2}\right) \frac{(g_\xi - g_\beta) \sqrt{\xi^2 + D_0^2}}{\sqrt{\xi^2 + D_0^2}} \frac{1}{D_0 \left(1 + \frac{\xi^2}{D_0^2}\right)} d\xi
$$

$$
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_0 \left(1 + \frac{\xi^2}{D_0^2}\right) \frac{(g_\xi - g_\beta) \sqrt{\xi^2 + D_0^2}}{\sqrt{\xi^2 + D_0^2}} d\xi
$$

(54)

This clearly demonstrates that $\hat{g}(\alpha, \beta)$ is a better representation since Equation (53) is a convolution while Equation (54) is not. \qed
2.3 Convolution in Fan Beam Geometry

Proposition 3. Let $\mu^*(x, y) = \mu(x, y) * h(x, y)$. Then,

$$\hat{g}_{\mu^*}(\alpha, \beta) = D_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \hat{g}_{\mu}(\tilde{\alpha}, \beta + \alpha - \tilde{\alpha}) \hat{g}_h(\tilde{\alpha}, \beta + \alpha - \tilde{\alpha}) \cos \tilde{\alpha} d\tilde{\alpha},$$

(55)

where $\tilde{\alpha} = \sin^{-1}(\sin \alpha - \sin \tilde{\alpha})$.

Proof. We have from Corollary 1, Theorem 2, letting $\xi = D_0 \sin \alpha$ and $\tilde{\xi} = D_0 \sin \tilde{\alpha}$ in Figure(10) that

$$\hat{g}_{\mu^*}(\alpha, \beta) = \rho_{\mu^*}(D_0 \sin \alpha, \beta + \alpha)$$

$$= \int_{-\infty}^{\infty} \rho_{\mu}(\tilde{\xi}, \beta + \alpha) \rho_h(D_0 \sin \alpha - \tilde{\xi}, \beta + \alpha) d\tilde{\xi}$$

$$= \int_{-\infty}^{\infty} \rho_{\mu}(D_0 \sin \tilde{\alpha}, \beta + \alpha) \rho_h(D_0 \sin \alpha - D_0 \sin \tilde{\alpha}, \beta + \alpha) D_0 \cos \tilde{\alpha} d\tilde{\alpha}$$

$$= \int_{-\infty}^{\infty} \hat{g}_{\mu}(\tilde{\alpha}, \beta + \alpha - \tilde{\alpha}) \rho_h(D_0 \sin \tilde{\alpha}, \beta + \alpha) D_0 \cos \tilde{\alpha} d\tilde{\alpha}$$

$$= \int_{-\infty}^{\infty} \hat{g}_{\mu}(\tilde{\alpha}, \beta + \alpha - \tilde{\alpha}) g_h(\tilde{\alpha}, \beta + \alpha - \tilde{\alpha}) D_0 \cos \tilde{\alpha} d\tilde{\alpha}.$$

Corollary 4. Filtering in the image space is equivalent to separately transforming each line $\alpha + \beta = c_0$ in the fan beam sinogram $g_\mu(\alpha, \beta)$ to $\hat{g}_{\mu^*}(\alpha, \beta)$, Figure(12), via Equation(55).

Proof. Consider a point $(\alpha_1, \beta_1)$ on the line $\alpha + \beta = c_0$. Then $\hat{g}_{\mu^*}$ requires $\hat{g}_{\mu}(\tilde{\alpha}, \beta + \alpha_1 - \tilde{\alpha})$ for all $\tilde{\alpha}$, namely, $\hat{g}_{\mu}$ for all points of the line $\alpha + \beta = c_0$. It also requires $\hat{g}_h(\tilde{\alpha}, \beta + \alpha - \tilde{\alpha})$ for all $\tilde{\alpha}$, again pointing to the same line. Since no other information is required, the filtering operation transforms a line of the sinogram to the same line.

Remark: The transformation is local due to the local extent of the filter $h(x, y)$ which implies that $\hat{g}_h(\alpha, \beta)$ has limited extent.

Example: (Gaussian filter in the fan beam geometry) According to Corollary(5), the Radon Transform of a Gaussian filter $h(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$ is $\rho_h(\xi, \theta) = \frac{1}{\sqrt{2\pi} e^{-\frac{\xi^2}{2\sigma^2}}}$. Thus, by Corollary(1)

$$\hat{g}_h(\alpha, \beta) = \rho_h(D_0 \sin \alpha, \beta + \alpha)$$

$$= \frac{1}{\sqrt{2\pi} e^{-\frac{D_0^2 \sin^2 \alpha}{2\sigma^2}}}.$$
Figure 12: 2D spatial filtering becomes a 1D transform in the projection data in fan beam geometry. The sinogram $\hat{g}_{\mu^*}$ of the filtered image $\mu^* = \mu * h$ can be obtained by transforming each line $\alpha + \beta = c_0$ separately from other points.

Similarly, for a shifted filter $\hat{h}(x, y) = \frac{1}{2\pi \sigma^2} e^{-\frac{(x-x_0)^2+(y-y_0)^2}{2\sigma^2}}, \rho_h = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\xi-x_0 \cos \theta - y_0 \sin \theta)^2}{2\sigma^2}}$.

Thus,

$$\hat{g}_h = \rho_h(D_0 \sin \alpha, \beta + \alpha) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(D_0 \sin \alpha - x_0 \cos(\beta + \alpha) - y_0 \sin(\beta + \alpha))^2}{2\sigma^2}}$$

Figure(13) shows the sinogram for the filter $\hat{h}$ for several selections of $(x_0, y_0) \in \{(0, 0), (50, 0), (0, 50), (50, 50)\}$.

Figure 13:
3 Flirting with Filtering

3.1 Introduction

The idea that structure in the image space can be more effectively detected when working directly with view data is based on a number of intuitions. First, the inherent discretization grid arises from the measurement process itself where a discrete set of views (orientations) are taken using a discrete set of detectors, while the image space discretization is artificially introduced to enable computation and visualization in the physically meaningful domain. Thus, in cases where improved resolution is of crucial importance, it would be wise to re-consider computations directly in the view space. Second, a general expectation is that since image space filtering and reconstruction from view data are both linear operations, perhaps filtering in the image space can be viewed as filtering in the view space. Specifically, matched filtering for detecting particular structures can be achieved by considering a filter projected in the view space and doing the matched filtering there. Thus, the current algorithm projects the matched filter $h(x, y)$ centered at $(x_0, y_0)$ to the view space and considers the inner product of this view space representation of the filter, $\rho_{h,x_0,y_0}(\xi, \theta)$ and the view space representation of the image $\mu(x, y)$, namely $\rho_{\mu}(\xi, \theta)$ in search of products exceeding a predetermined threshold $\tau$:

$$\text{structure at } (x_0, y_0) \text{ iff } \int_0^{2\pi} \int_{-\infty}^{\infty} \rho_{\mu}(\xi, \theta)\rho_{h,x_0,y_0}(\xi, \theta) d\xi d\theta > \tau.$$  

(56)

In this chapter, we evaluate the validity of this algorithm and examine matched filtering in the image space:

$$\text{structure at } (x_0, y_0) \text{ iff } \mu^*(x_0, y_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x, y)h(x-x_0, y-y_0)dxdy > \tau_{\mu}$$  

(57)

in regard to an equivalent operation in the view space.

3.2 Filtering

Proposition 4. Let $\hat{h}(x, y) = h(x - x_0, y - y_0)$. Then $\rho_{h}(\xi, \theta) = \rho_{h}(\xi - x_0 \cos \theta - y_0 \sin \theta, \theta)$, where $\rho_{h}$ and $\rho_{\hat{h}}$ are the view space representations of $h$ and $\hat{h}$, respectively. If $(r_0, \phi_0)$ is the polar representation of $(x_0, y_0)$, then $\rho_{h}(\xi, \theta) = \rho_{h}(\xi - r_0 \cos(\theta - \phi), \theta)$.

Proof. Let

$$\begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}.$$
Then,

\[ \rho_h(\xi, \theta) = \int_{-\infty}^{\infty} \hat{h}(\xi \cos \theta - \eta \sin \theta, \xi \sin \theta + \eta \cos \theta) d\eta \]

\[ = \int_{-\infty}^{\infty} \hat{h}(\xi \cos \theta - \eta \sin \theta - x_0, \xi \sin \theta + \eta \cos \theta - y_0) d\eta \]

\[ = \int_{-\infty}^{\infty} h((\xi - \xi_0) \cos \theta - (\eta - \eta_0) \sin \theta, (\xi - \xi_0) \sin \theta + (\eta - \eta_0) \cos \theta) d\eta \]

\[ = \int_{-\infty}^{\infty} h((\xi - \xi_0) \cos \theta - \eta \sin \theta, (\xi - \xi_0) \sin \theta + \eta \cos \theta) d\eta \]

\[ = \rho_h(\xi - \xi_0, \theta) \]

\[ = \rho_h(\xi - x_0 \cos \theta - y_0 \sin \theta, \theta). \]

...
have

\[ \rho_{\mu^*}(\xi, \theta) = \int_{-\infty}^{\infty} \mu^*(\xi \cos \theta - \eta \sin \theta, \xi \sin \theta + \eta \cos \theta) d\eta \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(\xi \cos \theta - \eta \sin \theta - u, \xi \sin \theta + \eta \cos \theta - v) h(u, v) \, du \, dv \, d\eta \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \mu(\xi \cos \theta - \eta \sin \theta - u, \xi \sin \theta + \eta \cos \theta - v) \, d\eta \right] h(u, v) \, du \, dv \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(\xi \cos \theta - \eta \sin \theta - \xi \cos \theta + \eta \sin \theta, \xi \sin \theta + \eta \cos \theta) \, d\eta \, h(u, v) \, du \, dv \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(\xi \cos \theta - \eta \sin \theta - \xi \cos \theta + \eta \sin \theta, \xi \sin \theta + \eta \cos \theta) \, d\eta \, h(u, v) \, du \, dv \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(\xi - \eta \sin \theta - \xi \cos \theta + \eta \cos \theta) \, d\eta \, h(u, v) \, du \, dv \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(\xi - \eta \sin \theta - \xi \cos \theta + \eta \cos \theta) \, d\eta \, h(u, v) \, du \, dv \]

\[ = \rho_{\mu}(\xi, \theta) *_{\xi} \rho_{\mu}(\xi, \theta). \]

In other words, to convolve a filter \( h \) with image \( \mu \), we can equivalently convolve each orientation column \( \theta \) of the sinogram \( \rho_{\mu} \) with the respective column of the sinogram \( \rho_{h} \) to reconstruct \( \rho_{\mu^*} \) at that column.

**Remark:** This approach can be used to reconstruct the original data. Observe that when the filter \( h \) is an impulse, i.e. \( h(x, y) = \delta(\sqrt{x^2 + y^2}) \), we have

\[ \rho_{h}(\xi, \theta) = \int_{-\infty}^{\infty} h(\xi \cos \theta - \eta \sin \theta, \xi \sin \theta + \eta \cos \theta) \, d\eta = \int_{-\infty}^{\infty} \delta(\sqrt{\xi^2 + \eta^2}) \, d\eta = \delta(\xi). \]

Since \( \rho_{h}(\omega, \theta) = 1 \), \( \rho_{h}^*(\xi, \theta) = F^{-1}(|\omega|) \), thus giving the filtered backprojection process. However, while \( |\omega| \) is not well-behaved in the frequency domain in the case of \( h \) being an impulse, in the general case \( \rho_{h} \) can be well-behaved!!! For example, when \( h \) has a Gaussian drop-off, the rising effects of \( |\omega| \) is curtailed. In summary, general purpose reconstruction probes the
image with an *impulse filter* to generate image value at pixels, and only after
reconstruction is the existence of specific structures probed. However, while
the general reconstruction problem is ill-conditioned, the direct probing of
specific structures in the view space may not be!!!

**Proposition 5** (Radially Symmetric Filters). The Radon Transform of a
radially symmetric filter \( h(x, y) = H(\sqrt{x^2 + y^2}) \) centered at \((x_0, y_0)\) is

\[
\rho_h(\xi, \theta) = \int_{-\infty}^{\infty} H(\sqrt{(\xi - (x_0 \cos \theta - y_0 \sin \theta))^2 + \eta^2}) d\eta.
\]

**Proof.** The Radon Transform of \( h(x, y) \) centered at the origin is

\[
\rho_h(\xi, \theta) = \int_{-\infty}^{\infty} h(\xi \cos \theta - \eta \sin \theta, \xi \sin \theta + \eta \cos \theta) d\eta
\]

\[
= \int_{-\infty}^{\infty} H(\sqrt{\xi^2 + \eta^2}) d\eta.
\]

Now, for \( \hat{h}(x, y) = h(x - x_0, y - y_0) \), we have by Proposition(4)

\[
\rho_{\hat{h}}(\xi, \theta) = \rho_h(\xi - (x_0 \cos \theta - y_0 \sin \theta), \theta),
\]

and the proposition follows.

**Corollary 5** (Radon Transform of a Gaussian Filter). The Radon Transform
of a Gaussian filter where centered at \((x_0, y_0)\), i.e., \( h(x, y) = h(x - x_0, y - y_0) \)

\[
h(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2 + y^2)}{2\sigma^2}},
\]

is

\[
\rho_h(\xi, \theta) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\xi - x_0 \cos \theta - y_0 \sin \theta)^2}{2\sigma^2}}.
\]

**Proof.** We have

\[
\rho_h(\xi, \theta) = \int_{-\infty}^{\infty} h(\xi \cos \theta - \eta \sin \theta, \xi \sin \theta + \eta \cos \theta) d\eta
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{(\xi^2 + \eta^2)}{2\sigma^2}} d\eta
\]

\[
= \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{\xi^2}{2\sigma^2}}.
\]

(59)

Now, by Proposition(5)

\[
\rho_{\hat{h}}(\xi, \theta) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\xi - x_0 \cos \theta - y_0 \sin \theta)^2}{2\sigma^2}}.
\]


This implies that a 2D Gaussian spot in the image space maps to a collection of 1D Gaussians of the same spread on each column of the view space, but whose center $\xi_0 = x_0 \cos \theta + y_0 \sin \theta = r_0 \cos(\theta - \phi_0)$ is on a sinusoid with respect to $\theta$.

### 3.3 Matched Filtering

**Theorem 3** (Matched filtering in the view space). *A structure is present at the image indicated by image space matched filtering\(^{60}\)

$$\mu^*(x_0, y_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x, y) h(x - x_0, y - y_0) dx dy > \tau_\mu \quad (60)$$

if and only if

$$\int_0^{2\pi} [\rho_\mu(\xi, \theta) \ast_{\xi} \nu_h(\xi, \theta)] (x_0 \cos \theta + y_0 \sin \theta) d\theta > 2\pi \tau_\mu, \quad (61)$$

where $\nu_h$ is the view space matched filter, the projected filter, defined as

$$\nu_h(\xi, \theta) = \mathcal{F}_\omega^{-1} \{ \overline{\rho_h(\omega, \theta)} | \omega \}. \quad (62)$$

where $\overline{\rho_h(\omega, \theta)} = \mathcal{F}_\xi \{ \rho_h(\xi, \theta) \}(\omega)$. (bar denotes the Fourier Transform, i.e., \( \mathcal{F}(\omega) = \mathcal{F}\{f(x)\} \).)

**Proof.** We have $\overline{\mu^*} = \overline{\mu h}$. Recall also that by the Central Slice Theorem, $\overline{\rho}(\omega, \theta)$ is the Fourier Transform of $\rho_\mu(\xi, \theta)$ with $\xi$ as the free variable and $\theta$ as a parameter, and similarly for $h$,

$$\begin{align*}
\overline{\overline{\rho}(\omega, \theta)} &= \mathcal{F}_\xi \{ \rho_\mu(\xi, \theta) \}(\omega) = \overline{\rho_\mu(\omega, \theta)} \\
\overline{\overline{h}(\omega, \theta)} &= \mathcal{F}_\xi \{ \rho_h(\xi, \theta) \}(\omega) = \overline{\rho_h(\omega, \theta)}. \quad (63)
\end{align*}$$
Then, we can rewrite $\mu^*$ in the Fourier domain

$$
\mu^*(x_0, y_0) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \hat{\mu}(\omega, \theta) e^{i\omega(x_0 \cos \theta + y_0 \sin \theta)} |\omega| d\omega d\theta
$$

(64)

$$
= \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \overline{\mu}(\omega, \theta) \overline{h}(\omega, \theta) e^{i\omega(x_0 \cos \theta + y_0 \sin \theta)} |\omega| d\omega d\theta
$$

$$
= \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} P_\mu(\omega, \theta) \overline{h}(\omega, \theta) e^{i\omega(x_0 \cos \theta + y_0 \sin \theta)} |\omega| d\omega d\theta
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} P_\mu(\omega, \theta) \overline{h}(\omega, \theta) |\omega| e^{i\omega(x_0 \cos \theta + y_0 \sin \theta)} d\omega \right] d\theta
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} P_\mu(\omega, \theta) \overline{h}(\omega, \theta) e^{i\omega(x_0 \cos \theta + y_0 \sin \theta)} d\omega \right] d\theta
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}^{-1} [P_\mu(\omega, \theta) \overline{h}(\omega, \theta)] (x_0 \cos \theta + y_0 \sin \theta) d\theta
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} [\rho_\mu(\xi, \theta) * \nu^*_h(\xi, \theta)] (x_0 \cos \theta + y_0 \sin \theta) d\theta. \quad (65)
$$

Observe that matched filtering thresholds in the view space and image space are now related.

### 3.4 Detecting Spots

#### 3.4.1 Detecting A Gaussian Spot

We now attempt to analytically derive an expression for the view space matched filter $\rho^*_h(\xi, \theta)$ for the specific case of the Gaussian spot. Consider a matched filter

$$
h(x, y) = \frac{1}{2\pi\sigma_1^2} e^{-\frac{x^2}{2\sigma_1^2}} - \frac{1}{2\pi\sigma_2^2} e^{-\frac{y^2}{2\sigma_2^2}}, \quad \text{where} \quad \sigma_2 > \sigma_1, \quad (66)
$$

where $\sigma_1$ is the size of the Gaussian spot. Then, by additivity,

$$
\rho_h(\xi, \theta) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{\xi^2}{2\sigma_1^2}} - \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{\xi^2}{2\sigma_2^2}}, \quad (67)
$$

Then, as shown in Appendix(A.2)

$$
\overline{\rho}_h(\xi, \theta) = \mathcal{F}_\xi \{ \rho_h(\xi, \theta) \} = e^{-\frac{\xi^2}{2\sigma_1^2}} - e^{-\frac{\xi^2}{2\sigma_2^2}}. \quad (68)
$$
Then,
\[
\tilde{p}_h^\ast(\omega, \theta) = |\omega| e^{-\frac{\omega^2 \sigma_1^2}{2}} - |\omega| e^{-\frac{\omega^2 \sigma_2^2}{2}}.
\] (69)

or
\[
\rho_h^\ast(\xi, \theta) = \mathcal{F}^{-1}\{|\omega| e^{-\frac{\omega^2 \sigma_1^2}{2}} - |\omega| e^{-\frac{\omega^2 \sigma_2^2}{2}}\}.
\] (70)

Using Equation (110) in Appendix (A.4), we have
\[
\rho_h^\ast(\xi, \theta) = \frac{1}{\pi \sigma_1^2} - \frac{\sqrt{2}}{\pi \sigma_1^4} \xi e^{-\frac{\xi^2 \sigma_1^2}{2}} \int_0^{\frac{\sqrt{2} \xi}{\sigma_1}} e^{\zeta^2} d\zeta - \frac{1}{\pi \sigma_2^2} + \frac{\sqrt{2}}{\pi \sigma_2^4} \xi e^{-\frac{\xi^2 \sigma_2^2}{2}} \int_0^{\frac{\sqrt{2} \xi}{\sigma_2}} e^{\zeta^2} d\zeta.
\] (71)

Figure (14) shows examples of the filter \( h \) and the projected filter \( \nu_h \) for several values of \( \sigma \).

![Figure 14](image)

Figure 14: First row figures show the profile of \( h(x, y) \) along a line passing through the origin for \( \sigma_1 = 0.5, 1 \) and 2 respectively. Second row figures show \( \rho_h^\ast(\xi, \theta) \) for \( \sigma_1 = 0.5, 1 \) and 2 respectively.

### 3.4.2 Detecting A Circular Spot

A circular spot can be represented by its characteristic function:

\[
\mu_R(x, y) = \begin{cases} 
1 & x^2 + y^2 \leq R^2 \\
0 & x^2 + y^2 > R^2.
\end{cases}
\]
The Radon Transform of $\mu_R$ can be easily computed using $(\xi \cos \theta - \eta \sin \theta)^2 + (\xi \sin \theta + \eta \cos \theta)^2 = \xi^2 + \eta^2$

$$\rho_{\mu_R}(\xi, \theta) = \int_{-\infty}^{\infty} \mu_R(\xi \cos \theta - \eta \sin \theta, \xi \sin \theta + \eta \cos \theta) d\eta$$

$$= \begin{cases} \int_{\sqrt{R^2 - \xi^2}}^{\infty} d\eta = 2\sqrt{R^2 - \xi^2} & \text{if } \xi < R \\ \int_{-\sqrt{R^2 - \xi^2}}^{\infty} d\eta = \sqrt{R^2 - \xi^2} & \text{if } \xi \geq R. \end{cases}$$

Now for detecting this structure using a center-surround matched filter, we use $h(x, y) = \mu_R(x, y) - A\mu_{\bar{R}}(x, y)$, when $\bar{R} > R$ and $\bar{A}$ is chosen to give zero area. Since the area under $\mu_R$ is $\pi R^2$, we need $\pi R^2 - A\pi R^2 = 0$, or $A = \frac{R^2}{\bar{R}}$, so that a matched filter for a circular spot is

$$h(x, y) = \mu_R(x, y) - \frac{R^2}{\bar{R}} \mu_{\bar{R}}(x, y). \quad (72)$$

In our previous experiments, the area of the center and surround have also been matched, i.e. $\pi R^2 = \pi \bar{R}^2 - \pi R^2$, or $\bar{R} = \sqrt{2}R$. Then, the Radon Transform of the matched filter (72) is [Was there a reason for this?]

$$\rho_h(\xi, \theta) = 2\sqrt{R^2 - \xi^2} - 2\frac{R^2}{\bar{R}} \sqrt{\bar{R}^2 - \xi^2}$$

$$= \begin{cases} 2\sqrt{R^2 - \xi^2} - 2\frac{R^2}{\bar{R}} \sqrt{\bar{R}^2 - \xi^2} & \text{if } 0 < \xi < R \\ - 2\frac{R^2}{\bar{R}} \sqrt{\bar{R}^2 - \xi^2} & \text{if } R < \xi < \bar{R} \\ 0 & \text{if } \bar{R} < \xi \end{cases} \quad (73)$$

As shown in Appendix(B.1),

$$\mathcal{F}_x(\sqrt{A^2 - x^2}) = \frac{\pi A}{\omega} \left( \frac{\sin(\omega A)}{(\omega A)^2} - \frac{\cos(\omega A)}{\omega A} \right).$$

Using $A = R$ and $x = \xi$, we get

$$\mathcal{F}_\xi(\sqrt{R^2 - \xi^2}) = \frac{\pi R}{\omega} \left( \frac{\sin(\omega R)}{(\omega R)^2} - \frac{\cos(\omega R)}{\omega R} \right).$$

Hence,

$$\mathcal{F}_\xi(\rho_h(\xi, \theta)) = 2 \left[ \frac{\pi R}{\omega} \left( \frac{\sin(\omega R)}{(\omega R)^2} - \frac{\cos(\omega R)}{\omega R} \right) \right] - 2\frac{R^2}{\bar{R}} \left[ \frac{\pi \bar{R}}{\omega} \left( \frac{\sin(\omega \bar{R})}{(\omega \bar{R})^2} - \frac{\cos(\omega \bar{R})}{\omega \bar{R}} \right) \right].$$

39
and

\[ F_{\xi}(\rho_{h}(\xi, \theta)) = 2 \left[ |\omega| \frac{\pi R}{\omega} \left( \frac{\sin(\omega R)}{(\omega R)^2} - \frac{\cos(\omega R)}{\omega R} \right) \right] - 2 \frac{R^2}{R} \left[ |\omega| \frac{\pi R}{\omega} \left( \frac{\sin(\omega R)}{(\omega R)^2} - \frac{\cos(\omega R)}{\omega R} \right) \right] \]

or

\[ \rho_{h}(\xi, \theta) = 2 F^{-1} \left\{ |\omega| \frac{\pi R}{\omega} \left( \frac{\sin(\omega R)}{(\omega R)^2} - \frac{\cos(\omega R)}{\omega R} \right) \right\} - 2 \frac{R^2}{R} F^{-1} \left\{ |\omega| \frac{\pi R}{\omega} \left( \frac{\sin(\omega R)}{(\omega R)^2} - \frac{\cos(\omega R)}{\omega R} \right) \right\} . \]

Figure 15: First row figures show the profile of \( h(x, y) \) along a line passing through the origin for \( R = 1, 2 \) and 4 respectively. Second row figures show \( \nu_{h}(\xi, \theta) \) for \( R = 1, 2 \) and 4 respectively.

As shown in Appendix (B.2),

\[ F^{-1} \left\{ |\omega| \frac{\pi R}{\omega} \left( \frac{\sin(\omega R)}{(\omega R)^2} - \frac{\cos(\omega R)}{\omega R} \right) \right\} = 1 + \frac{\xi}{2R} \ln \left| \frac{\xi - R}{\xi + R} \right|. \]

Thus,

\[ \rho_{h}(\xi, \theta) = 2 \left[ 1 + \frac{\xi}{2R} \ln |\xi - R| \right] - 2 \frac{R^2}{R} \left[ 1 + \frac{\xi}{2R} \ln \left| \frac{\xi - R}{\xi + R} \right| \right] . \]  

Figure (15) shows examples of the filter \( h \) and the projected filter \( \nu_{h} \) for several values of \( R \).
4 Taylor Expansion in the View Space

5 Geometric Tomography

Assume the image $\mu(x,y)$ is made of several smooth regions of smoothly varying intensity such as the phantom shown in Figure(16a), i.e. that $\mu$ is piecewise smooth, with smooth contours of discontinuity. Consider a particular projection angle $\theta$ and some $\xi$ along the detector array. In general, this ray goes through a number of regions $R_i$, but is not tangent to them. In some cases the ray would be tangent to a region $R_i$. In general, $\rho$ is continuous as a function of $\xi$, for each $\theta$.

Figure 16: (a) The image $\mu(x,y)$ is assumed to be piecewise smooth. (b) The Radon Transform of $\mu$. (c) Edges extracted from the Radon Transform image (shown in green). A hand traced contour is shown in red. (d) The red contour is backprojected on the original image.
When the ray is not tangent to any region, it integrates through the regions it intersects with. In this case, the continuity of the image and the region implies that $p_\xi$ is smooth. When the ray is tangent to a region, however, as in the blue ray shown in Figure(16a), the first derivative of the Radon transform is no longer continuous since it is determined on each side by two distinct regions, one on the left ($\xi^-$) and one on the right side ($\xi^+$) of $\xi$. In the rare case of multiple tangent rays, the same statement holds. We have informally shown:

**Remark:** When the image $\mu(x, y)$ is piecewise smooth with smooth discontinuity curves, the derivative if the Radon Transform $p_\xi$ is discontinuous if and only if the projection ray is tangent to at least one image discontinuity.

This implies that edges detected on the projection image point to edges in the image, although they are not enough to localize these edges in space, Figure(16b). While this localization is not possible from a single view $\theta$, it may be possible to integrate multiple views to achieve this. Unfortunately, the rays in all other views passing through the edge of interest are transversal to the edge and thus no discontinuities can arise! The information about each edge is fully focused on a single view. What is worse is that if due to discretization, this view is missed, so is all the information pertaining to that edge!

A second constraint pertaining to X-ray images of realistic objects such as those obtained from anatomy is that not only the images are piecewise smooth, but also the discontinuity curves themselves are piecewise smooth. What this implies is that while for each image edge information can be obtained from only one discontinuity point in the Radon Transform, through the continuity of the edge on a curve we will have access to the Radon Transform discontinuities of its neighboring points!

Formally, consider a curve $C_\mu(s)$ seperating two distinct regions in the $\mu$ image plane. Let the tangent and normal to this curve be denoted as $T_\mu(s)$ and $N_\mu(s)$, respectively. For each point $C_\mu(s)$, there is a discontinuity in $p_\xi(\xi, \theta)$ where $\theta$ denotes the orientation of the detector array normal to $T_\mu(s)$, and $\xi$ id the projection of $C_\mu(s)$ on that view, Figure(??). As we traverse $C_\mu(s)$, this traces out a curve in the sinogram space, Figure(??), denoted by $C_\rho(s) = (\xi(s), \theta(s))$. Along the curve $C_\rho(s)$ in the view (sinogram) space, $\rho$ has discontinuous first derivatives.

**Proposition 6.** Let $\rho(\xi, \theta)$ be the Radon Transform of $\mu(x, y)$ containing a
Figure 17: (a) The discontinuity $\bar{A}$ at $\xi_1$ detected in the Radon Transform derivative corresponding to angle $\theta$, predicts a discontinuity $A$ in the image domain, (b) Similarly $\bar{B}$ at $\xi_2$ points to the existance of an edge $B$ in the projection view $\theta + d\theta$, (c) The intersection point $P$ of the two rays localizes $A$ and $B$ in the limit as $d\theta \rightarrow 0$.

A smooth curve $C_\mu(s)$ separating two distinct regions, which traces out a curve $C_\rho(s) = (\xi(s), \theta(s))$ in the view space along which $\rho_\xi$ is discontinuous. Then, $C_\mu(s)$ can be obtained for $C_\rho(s)$ as

$$C_\mu(s) = \xi(s)(\cos \theta(s), \sin \theta(s)) + \frac{d\xi}{ds}(-\sin \theta(s), \cos \theta(s)).$$  \hspace{1cm} (75)

Proof. For each point of $C_\rho(s) = (\xi(s), \theta(s))$, the point $C_\mu(s)$ is located on the line orthogonal to the detector array, $\xi$-axis, in the image domain, at $\xi$, Figure(18). Let $\eta(s)$ denote the distance between $C_\mu(s)$ and the detector array. Then

$$C_\mu(s) = \xi(s)(\cos \theta, \sin \theta) + \eta(s)(-\sin \theta, \cos \theta).$$  \hspace{1cm} (76)
We also know that the tangent to $C_\mu(s)$ is orthogonal to the $\xi$-axis

$$\frac{d}{ds}C_\mu(s) = g_\mu(s)(-\sin \theta, \cos \theta), \quad (77)$$

where $g_\mu(s)$ is the metric of parametrization with respect to $s$. Differentiating both sides of Equation (76)

$$\frac{d}{ds}C_\mu(s) = \frac{d\xi}{ds}(\cos \theta, \sin \theta) + \xi(s)(-\sin \theta, \cos \theta) \frac{d\theta}{ds} +$$

$$\frac{d\eta}{ds} (-\sin \theta, \cos \theta) + \eta(s)(-\cos \theta, -\sin \theta) \frac{d\theta}{ds} \quad (78)$$

$$= \left( \frac{d\xi}{ds} - \eta(s) \frac{d\theta}{ds} \right) (\cos \theta, \sin \theta) + \left( \xi(s) \frac{d\theta}{ds} + \frac{d\eta}{ds} \right) (-\sin \theta, \cos \theta).$$

Figure 18:
Figure 19:

Equating Equation (77) and (78), we have

\[
\begin{cases}
\frac{d\xi}{ds} - \eta(s) \frac{d\theta}{ds} = 0 \\
\xi(s) \frac{d\theta}{ds} + \frac{d\eta}{ds} = g_\mu(s).
\end{cases}
\] (79)

The first equation leads to a solution for \( \eta(s) \)

\[ \eta(s) = \frac{\frac{d\xi}{ds}}{\frac{d\theta}{ds}} = \frac{d\xi}{d\theta}, \] (80)

then leading to

\[ C_\mu(s) = \xi(s)(\cos \theta(s), \sin \theta(s)) + \frac{\frac{d\xi}{ds}}{\frac{d\theta}{ds}}(-\sin \theta(s), \cos \theta(s)). \] (81)

Let \( \frac{d}{ds}C_\rho(s) = g_\rho(s)\tau_\rho(s) \), where the tangent vector to \( C_\rho(s) \) is represented as \( \tau_\rho(s) = (\cos \phi(s), \sin \phi(s)) \). Then, \( \frac{d\xi}{ds} = g_\rho(s) \cos \phi(s) \) and \( \frac{d\eta}{ds} = g_\rho(s) \sin \phi(s) \), so that

\[ \frac{d\xi}{d\theta} = \frac{g_\rho(s) \cos \phi(s)}{g_\rho(s) \sin \phi(s)} = \cot \phi(s). \] (82)
Note that the second equation in Equation (79) gives the relative speed(?) of parametrization at $\mu$ by differentiating the first equation and substituting in the second.

$$
\frac{d\eta}{ds} = \frac{d}{ds} \left( \frac{d\xi}{d\theta} \right) = -\frac{d\phi(s)}{\sin^2 \phi} = -\frac{g_\rho(s)k_\rho(s)}{\sin^2 \phi},
$$

so that

$$
g_\mu(s) = \xi(s) \frac{d\theta}{ds} - \frac{g_\rho(s)k_\rho(s)}{\sin^2 \phi}.
$$

**Corollary 6.** The edge map of $\mu(x, y)$ can be obtained by Equation (75) from the edge map of $\rho_\xi(\xi, \theta)$.

**PLACE-HOLDER**

Figure 20: Can to put the simulation results using Proposition 6 here...

### 5.1 Fan Beam Geometry

The derivation for the fan beam geometry is similar. Consider Figure (21) and let the source $S$ be at distances $D_0$ and $D$ away from the real detector array and the origin, respectively. Let the sinogram be specified in the normalized detector array passing through the origin. Assume $S$ rotates around $O$ and consider view at angle $\theta$. Let $\alpha = \angle(SO, SP)$ denote the angle of the ray tangent to the curve $C_\mu$ at $P$. Then

$$
C_\mu(s) = \overrightarrow{OA} + \overrightarrow{AP},
$$

where

$$
\overrightarrow{OA} = \xi(\cos \theta, \sin \theta),
$$

and

$$
\overrightarrow{AP} = \frac{\eta}{\cos \alpha} \left( \cos \left( \theta + \frac{\pi}{2} + \alpha \right), \sin \left( \theta + \frac{\pi}{2} + \alpha \right) \right) = \frac{\eta}{\cos \alpha} (-\sin(\theta+\alpha), \cos(\theta+\alpha)).
$$
We also have \( \xi = D \tan \alpha \), so that

\[
C_\mu(s) = D \tan \alpha (\cos \theta, \sin \theta) + \frac{\eta}{\cos \alpha} (-\sin(\theta + \alpha), \cos(\theta + \alpha)).
\]

On the other hand, the tangent to the curve \( C_\mu(s) \) is

\[
T_\mu(s) = -\frac{\overrightarrow{AP}}{|AP|} = g_\mu(s)(-\sin(\theta + \alpha), \cos(\theta + \alpha))
\]

On the other hand,

\[
\frac{dC_\mu(s)}{ds} = D(1 + \tan^2 \alpha) \frac{d\alpha}{ds} (\cos \theta, \sin \theta) + D \tan \alpha (-\sin \theta, \cos \theta) \frac{d\theta}{ds}
\]

\[
+ \frac{d\eta}{ds} \frac{1}{\cos \alpha} (-\sin(\theta + \alpha), \cos(\theta + \alpha)) + \frac{\eta \sin \alpha}{\cos^2 \alpha} \frac{d\alpha}{ds} (-\sin(\theta + \alpha), \cos(\theta + \alpha))
\]

\[
+ \frac{\eta}{\cos \alpha} (-\cos(\theta + \alpha), -\sin(\theta + \alpha)) \left( \frac{d\theta}{ds} + \frac{d\alpha}{ds} \right)
\]
Now, \(\frac{dC_{\mu}(s)}{ds} \cdot N_\mu = \frac{dC_{\mu}(s)}{ds}(\cos(\theta + \alpha), \sin(\theta + \alpha)) = 0\) so that after this dot product

\[
0 = D(1 + \tan^2 \alpha) \frac{d\alpha}{ds} (\cos \theta \cos(\theta + \alpha), \sin \theta \sin(\theta + \alpha)) + D \tan \alpha (-\sin \theta \cos(\theta + \alpha), \cos \theta \sin(\theta + \alpha)) \frac{d\theta}{ds} + \frac{\eta}{\cos \alpha} (-\cos^2(\theta + \alpha) - \sin^2(\theta + \alpha)) \left( \frac{d\theta}{ds} + \frac{d\alpha}{ds} \right)
\]

\[
= \frac{D}{\cos^2 \alpha} \frac{d\alpha}{ds} \cos \alpha + \frac{D \sin \alpha}{\cos \alpha} \sin \alpha \frac{d\theta}{ds} - \frac{\eta}{\cos \alpha} \left( \frac{d\theta}{ds} + \frac{d\alpha}{ds} \right)
\]

which gives

\[
\eta(s) = D \frac{d\alpha}{ds} + \frac{\sin^2 \alpha}{\cos^2 \alpha} \frac{d\theta}{ds}.
\]  

(83)

This can be written in terms of \(\xi\) and its derivatives using \(\xi = D \tan \alpha, \frac{d\xi}{ds} = D \frac{1}{\cos^2 \alpha} \frac{d\alpha}{ds} = D \frac{\xi^2 + \eta^2}{\eta^2} \frac{d\alpha}{ds} = \frac{D^2 + \xi^2}{\eta^2} \frac{d\alpha}{ds}\), and \(\sin \alpha = \frac{\xi^2}{D^2 + \xi^2}\), so that

\[
\eta(s) = D \frac{D \frac{d\xi}{ds} + \xi^2 \frac{d\alpha}{ds}}{D \frac{d\xi}{ds} + (D^2 + \xi^2) \frac{d\alpha}{ds}}
\]

\[
= D \frac{D \frac{d\xi}{ds} + \xi^2 \frac{d\alpha}{ds}}{D \frac{d\xi}{ds} + (D^2 + \xi^2) \frac{d\alpha}{ds}}.
\]

(84)

6 Cone Beam Tomography and Volumetric CT

In this section, we consider imaging a volume with an X-ray source and a planar detector. This is a natural progression from the evolution of single slice CT to spiral CT and then to multislice CT. As the number of slices increased dramatically from the initial 4 to 8, 16, 32, 128, the situation where an entire plane can be sensed simultaneously has become realistic, Figure(22). This is the case for \(\mu\)CT and the more experimental volumetric CT (e.g., Siemens) machines which use flat panel detectors. At a first glance, the tomographic reconstruction might appear to be one of collecting all line integrals through the volume and reconstructing the linear attenuation coefficient for the volume from these measurements. Note, however, that while the dimension of the unknowns is three, namely, \(\mu(x, y, z)\), the dimensionality of the measured data collected in this way is four, two for the detector \((u, v)\) and two for the position of the source, say as described by
points on a unit sphere. One can then speculate that not all line integrals may be needed, but rather a one-dimensional subset, where the source moves along some curve, is sufficient. This is indeed true: a one-dimensional set of measurements by moving the source along a curve (source orbit, or source trajectory) is shown to be sufficient if the trajectory is chosen appropriately. The proof, however, requires an organizing measurements of attenuation coefficients along integration planes, which defines the 3D Radon Transform, as opposed to integration lines which defines the cone beam transform. We define each below.

Figure 22: (a) The cone beam geometry: The detection array is some distance away from the object space origin $O$ which the source $S$ is a distance $D_0$ away and moves on a curve $\Gamma$. $SO$ is orthogonal to the detection array, which in analogy to the fan beam case is scaled to pass through origin (b).
Table 1: Table of Notations for the Cone Beam Geometry

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>X-ray source</td>
</tr>
<tr>
<td>( \Gamma(s) )</td>
<td>space curve describing the source orbit (trajectory) with ( s ) as length parameter</td>
</tr>
<tr>
<td>( O, \mathbf{i}, \mathbf{j}, \mathbf{k} )</td>
<td>object coordinate system</td>
</tr>
<tr>
<td>( D_0 )</td>
<td>the distance from ( S ) to origin ( O )</td>
</tr>
<tr>
<td>( D_1 )</td>
<td>the distance from the detection array to origin ( O )</td>
</tr>
<tr>
<td>( x, y, z )</td>
<td>coordinates in object coordinate system</td>
</tr>
<tr>
<td>( \mu )</td>
<td>linear attenuation coefficient of the object</td>
</tr>
<tr>
<td>( \tilde{O}, \mathbf{u}, \mathbf{v}, \mathbf{w} )</td>
<td>(virtual) detection coordinate system, ( \mathbf{w} = \frac{O_S}{</td>
</tr>
<tr>
<td>( \xi, \eta, \zeta )</td>
<td>coordinates in the detection coordinate system</td>
</tr>
<tr>
<td>( d )</td>
<td>distance of the integration plane from object space origin</td>
</tr>
<tr>
<td>( \mathbf{n} )</td>
<td>unit normal to the integration plane</td>
</tr>
<tr>
<td>( S, (\nu, \nu^\perp, n) )</td>
<td>integration plane coordinate system</td>
</tr>
<tr>
<td>( \mathcal{R}_\mu(d, \mathbf{n}) )</td>
<td>3D Radon Transform: integration along a plane of distance ( d ) from origin and normal ( \mathbf{n} )</td>
</tr>
<tr>
<td>( \mathcal{C}_\mu(s, \Theta) )</td>
<td>Cone Beam Transform: integration along a half-line from source ( S ) directed along ( \Theta )</td>
</tr>
</tbody>
</table>

### 6.1 Notation and Definition

Let the object coordinate system be described by coordinates \( (x, y, z) \) with \( O \) serving as the origin, Figure 22(a), and \( (\mathbf{i}, \mathbf{j}, \mathbf{k}) \) as unit vectors along the coordinates, respectively. The linear attenuation coefficient of the object is then described as \( \mu(x, y, z) \).

It is assumed that the source is at a fixed distance \( D_0 \) from this plane and that the closest point to \( S \) on the virtual detector array is \( \tilde{O} \). We assume that \( \tilde{O} = O \) (Ben, this may need to be relaxed). As in the case of fan beam geometry, it is easier to consider a virtual detector array, parallel to the actual detector array, that is parametrized along its own intrinsic coordinate system described by vectors \( (\mathbf{u}, \mathbf{v}) \), Figure 22(b). Define \( (\tilde{O}, \mathbf{u}, \mathbf{v}, \mathbf{w}) \) as the detection coordinate system. The source \( S \) is assumed to move along a curve \( \Gamma(s) \) referred to as the source orbit or source trajectory.

The notion of 3D Radon Transform requires integration of linear attenuation coefficients along planes passing through the source, each of which we refer to as an integration plane, see Table 1. An integration plane is specified by distance from origin \( O \) to the plane \( d \) and a unit normal \( \mathbf{n} \). Let \( M_0 \) be the closest point on this plane to the origin. Since the source \( S \) is by definition
on this plane, the vector $\overrightarrow{SM_0}$ when normalized defines a unique direction $\vec{\nu} = \frac{\overrightarrow{SM_0}}{|\overrightarrow{SM_0}|}$ which together with $\vec{\nu}^\perp = \vec{n} \times \vec{\nu}$ defines the integration plane coordinate system $(\vec{\nu}, \vec{\nu}^\perp, \vec{n})$, Figure 23.

Figure 23: An integration plane is specified in the object space by the distance $d$ of the integration plane from the origin and by the unit normal $\vec{n}$. The integration plane can be intrinsically described by a coordinate system centered at $S$ and with axis $\vec{\nu}, \vec{\nu}^\perp$ where $\vec{\nu} = \frac{\overrightarrow{SM_0}}{|\overrightarrow{SM_0}|}$, where $M_0$ is the closest point to the origin.

The vector $\vec{n}$ can be described in the spherical coordinate system of the object space as

$$\vec{n} = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi), \quad (85)$$

where $\phi = \angle(\vec{n}, \vec{k})$ and $\theta$ is the angle the projection of $\vec{n}$ onto the $xy$-plane makes with $\vec{i}$, Figure 23b. Similarly, in the detection space $\vec{n}$ can be expressed as

$$\vec{n} = (\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta). \quad (86)$$
Remark: We can express $\nu$ and $\nu^\perp$ as follows.

\[
\begin{align*}
\vec{\nu} &= \frac{(\overrightarrow{OS}, \vec{n})\vec{n} - \overrightarrow{OS}}{|(\overrightarrow{OS}, \vec{n})\vec{n} - \overrightarrow{OS}|} \\
&= \frac{d_0(\vec{w}, \vec{n})\vec{n} - d_0\vec{w}}{|d_0(\vec{w}, \vec{n})\vec{n} - d_0\vec{w}|} \\
&= \frac{d_0 \cos \beta (\cos \alpha \sin \beta \vec{u} + \sin \alpha \sin \beta \vec{v} + \cos \beta \vec{w}) - d_0 \vec{w}}{\sqrt{\sin^2 \beta \cos^2 \beta + \sin^4 \beta}} \\
&= (\cos \alpha \cos \beta \vec{u} + \sin \alpha \cos \beta \vec{v} - \sin \beta \vec{w}).
\end{align*}
\]

Similarly,

\[
\begin{align*}
\vec{\nu}^\perp &= \vec{n} \times \vec{\nu} \\
&= (\cos \alpha \sin \beta \vec{u} + \sin \alpha \sin \beta \vec{v} + \cos \beta \vec{w}) \times (\cos \alpha \cos \beta \vec{u} + \sin \alpha \cos \beta \vec{v} - \sin \beta \vec{w}) \\
&= (-\cos \alpha \sin^2 \beta - \cos \alpha \cos^2 \beta)(-\vec{v})(-\sin \alpha \sin^2 \beta - \sin \alpha \cos^2 \beta)(\vec{u}) \\
&= -\sin \alpha \vec{u} + \cos \alpha \vec{v}.
\end{align*}
\]

We are now ready to define the cone beam transform and the 3D Radon Transform.

**Definition 1.** The Cone Beam Transform of a scalar function $\mu(x, y, z)$ is defined as

\[
\begin{align*}
\mathcal{C}_\mu &: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \\
\mathcal{C}_\mu(S, \vec{\Theta}) &= \int_0^\infty \mu(\overrightarrow{OS} + r\vec{\Theta}) dr,
\end{align*}
\]

where $S$ is the source along its trajectory, and $\vec{\Theta}$ is an arbitrary unit vector.

One can extend the definition of the cone-beam transform from $\mathbb{R} \times \mathbb{R}^2$ to $\mathbb{R}^3 \times \mathbb{R}^3$ and by an abuse of notation use $\mathcal{C}_\mu$ to also denote the extended function

\[
\mathcal{C}_\mu(S, P) = \frac{1}{|P|} \mathcal{C}_\mu \left( S, \frac{P}{|P|} \right).
\]

Then the Fourier Transform of $\mathcal{C}_\mu$ is

\[
\overline{\mathcal{C}_\mu}(S, \omega) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{C}_\mu(S, P)e^{-i.S \cdot P} dP.
\]
The function \( \overline{C\mu} \) is homogeneous of degree -2, i.e.,

\[
\overline{C\mu}(S, \lambda \omega) = \int \int \int_{\mathbb{R}^3} C\mu(S, P)e^{-iP\lambda \omega}dP
\]

\[
= \frac{1}{\lambda^3} \int \int \int_{\mathbb{R}^3} C\mu(S, \frac{1}{\lambda} \tilde{P})e^{-i\tilde{P}\omega}d\tilde{P}
\]

\[
= \frac{1}{\lambda^3} \int \int \int_{\mathbb{R}^3} C\mu(S, \tilde{P})e^{-i\tilde{P}\omega}d\tilde{P}
\]

\[
= \frac{1}{\lambda^2} \overline{C\mu}(S, \omega).
\]

We follow [34] in their reformulation of Tuy’s inversion formula [29].

**Theorem 4** (Tuy’s Inversion Formula). Suppose Tuy's (modified) completeness condition is satisfied, i.e., that the set

\[
\Lambda(P, \Theta) = \{ s \in \mathbb{R}, (P - \Gamma(s)) \cdot \Theta = 0, \Gamma'(s) \cdot \Theta \neq 0, W_P(s, \Theta) \neq 0 \}
\]

is non-empty , where \( P \) is a point in space, \( P \in \mathbb{R}^3 \), \( \Theta \) is an arbitrary direction \( \Theta \in \mathbb{S}^2 \), and \( W_P(s, \Theta) \) is integrable with respect to its second variable satisfying

\[
\sum_{s \in \Lambda(P, \Theta)} W_P(s, \Theta) = 1 \quad \text{almost everywhere in } \Theta \in \mathbb{S}^2.
\]

Then

\[
\mu(P) = \frac{1}{(2\pi)^3} \int_{\mathbb{S}^2} \sum_{s \in \Lambda(P, \Theta)} \frac{W_P(s, \Theta)}{i\Gamma'(s) \cdot \Theta} \frac{\partial}{\partial t} \overline{C\mu}(\Gamma(t), \Theta) \bigg|_{t=s} d\Theta.
\]
Proof.

\[ \frac{\partial}{\partial t} C^\mu_\mu(\Gamma(t), \Theta) = \frac{\partial}{\partial t} \int \int \int_{\mathbb{R}^3} \mu(\Gamma(t), z) e^{-iz.\Theta} dz \]
\[ = \frac{\partial}{\partial t} \int \int \int_{\mathbb{R}^3} \left[ \int_0^\infty \mu(\Gamma(t) + rz) dr \right] e^{-iz.\Theta} dz \]
\[ = \frac{\partial}{\partial t} \int_0^\infty \int \int \int_{\mathbb{R}^3} \mu(W) e^{-i\frac{W-\Gamma(t)}{r} \Theta} \frac{1}{r^3} dW dr \]
\[ = \frac{\partial}{\partial t} \int_0^\infty \left[ \frac{1}{r^3} \int \int \int_{\mathbb{R}^3} \mu(W) e^{-i\frac{W}{r} \Theta} dW \right] e^{i\frac{1}{r} \Gamma(t).\Theta} dr \]
\[ = \frac{\partial}{\partial t} \int_0^\infty \hat{\mu}(\tilde{r}, \Theta) e^{i\tilde{r} \Gamma(t).\Theta} \tilde{r} d\tilde{r} \]
\[ = \left( \Gamma'(t), \Theta \right) \int_0^\infty \tilde{\mu}(r\Theta) e^{i\tilde{r} \Gamma(t)'\Theta} r^2 dr. \quad (91) \]

Next, To be continued

**Definition 2.** The 3D Radon Transform of a scalar function \( \mu(x, y, z) \) is defined as

\[ \mathcal{R}_\mu : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \]
\[ \mathcal{R}_\mu(d, \vec{n}) = \int \int_{\overrightarrow{OM}.\vec{n} = d} \mu(M) dM, \]

where \( M \) is a point on the integration plane defined by \( \overrightarrow{OM}.\vec{n} = d \) and \( dM \) is the area element.

### 6.2 Relating Cone Beam and Radon Transform

The following theorem from [20] asserts that the set of all integration plane measurements is sufficient to invert the transform.

**Theorem 5** (Inverse 3D Radon Transform[20]).

\[ \mu(M) = -\frac{1}{8\pi^2} \int \int_{n \in S^2} \frac{\partial^2 \mathcal{R}_\mu}{\partial d^2} (\overrightarrow{OM}.\vec{n}, \vec{n}) d\vec{n}. \]
Remark: The 3D Radon Transform can be expressed as
\[
\mathcal{R}_\mu(d, \vec{n}) = \int \int_{\overrightarrow{OМ}.n = d_0} \mu(M) dM
\]
\[
= \int_0^{2\pi} \int_0^\infty \mu(\overrightarrow{OS} + r \cos \gamma \vec{v} + r \sin \gamma \vec{v}_\perp) r dr d\gamma, \quad (92)
\]
where \( \vec{v} = \frac{(\overrightarrow{OS}.\vec{n} - \overrightarrow{OS})}{|\overrightarrow{OS}.\vec{n} - \overrightarrow{OS}|} \) and \( \vec{v}_\perp = \vec{n} \times \vec{v} \). This allows us to explore the relation between the cone beam transform and the 3D Radon Transform: When \( \Theta \) is restricted to the integration plane we have
\[
\mathcal{C}_\mu(S, \vec{\Theta}) = \int_0^\infty \mu(\overrightarrow{OS} + r \vec{\Theta}) dr
\]
\[
= \int_0^\infty \mu(\overrightarrow{OS} + r \cos \gamma \vec{v} + r \sin \gamma \vec{v}_\perp) dr.
\]
Thus, integrating all cone beam measurements along the integration plane gives
\[
\int_0^{2\pi} \mathcal{C}_\mu(s, \vec{\Theta}) d\gamma = \int_0^{2\pi} \int_0^\infty \mu(\overrightarrow{OS} + r \cos \gamma \vec{v} + r \sin \gamma \vec{v}_\perp) dr d\gamma \quad (93)
\]
Observe that this closely corresponds, but is not identical to the 3D Radon Transform in Equation(92). Since the inversion formula of Theorem 5 requires Equation(92) and the practical measurements are those of the cone beam transform, a way must be found to relate these two quantities. Grangeat’s insight [9, 10] was that by differentiating the cone beam transform with respect to \( \beta \), which is implicit in \( \vec{v} \), the \( r \) term can be brought out and the integrals can be related. The following theorem is based on Grangeat’s proof but is significantly shorter. This requires a weighted integral of cone beam rays along the integration plane.

**Definition 3.** For each point \( S \) and unit normal \( \vec{n} \) define the weighted cone beam integral.
\[
\mathcal{X}(S, \vec{n}) = \int_0^{2\pi} C_\mu(S, \vec{n}) \frac{1}{\cos \gamma} d\gamma = \int_0^{2\pi} \int_0^\infty \mu(\overrightarrow{OS} + r \cos \gamma \vec{v} + r \sin \gamma \vec{v}_\perp) \frac{1}{\cos \gamma} dr d\gamma \quad (94)
\]
where \( \gamma, \vec{v}, \) and \( \vec{v}_\perp \) are defined as before.

**Theorem 6** (Relating the Cone Beam and 3D Radon Transforms[9]).
\[
\frac{\partial}{\partial \beta} \mathcal{X}(S, \alpha, \beta) = \frac{\partial \mathcal{R}_\mu}{\partial d}(\overrightarrow{OS}, \vec{n}, \vec{n}).
\]
Proof. Observe from Equation 87 and 88 that $\vec{\nu}$ and $\vec{\nu}^\perp$ only depend on $\vec{n}(\alpha, \beta)$ but not on $d$. We therefore have

$$\begin{align*}
\frac{\partial \vec{\nu}}{\partial \beta} &= -\cos \alpha \sin \beta \vec{u} - \sin \alpha \sin \beta \vec{v} - \cos \beta \vec{w} = -\vec{n} \\
\frac{\partial \vec{\nu}^\perp}{\partial \beta} &= 0.
\end{align*}$$

Now, observe that for $\vec{\Theta}$ in the integration plane $\vec{\Theta} = r \cos \gamma \vec{\nu} + r \sin \gamma \vec{\nu}^\perp$ which implies that $\vec{\Theta}$ is implicitly a function of $\vec{n}(\alpha, \beta)$.

$$\frac{\partial \mu}{\partial \beta}(\overrightarrow{OS} + r \cos \gamma \vec{\nu} + r \sin \gamma \vec{\nu}^\perp) = \nabla \mu(\overrightarrow{OS} + r \cos \gamma \vec{\nu} + r \sin \gamma \vec{\nu}^\perp) r \cos \gamma \frac{\partial \vec{\nu}}{\partial \beta} = -r \cos \gamma \nabla \mu(\overrightarrow{OS} + r \cos \gamma \vec{\nu} + r \sin \gamma \vec{\nu}^\perp).\vec{n}. $$

This gives the $r$ term we needed so that as

$$\frac{\partial X}{\partial \beta}(s, \vec{n}) = \frac{\partial}{\partial \beta} \int_0^{2\pi} C_\mu(s, \vec{m}) \left( \frac{1}{\cos \gamma} \right) d\gamma = -\int_0^{2\pi} \int_0^\infty \nabla \mu(\overrightarrow{OS} + r \cos \gamma \vec{\nu} + r \sin \gamma \vec{\nu}^\perp) \vec{n} r dr d\gamma$$

On the other hand, differentiating the 3D Radon Transform with respect to $d$ gives

$$\frac{\partial R_\mu}{\partial d} = \int_0^{2\pi} \int_0^\infty \nabla \mu(\overrightarrow{OS} + r \cos \gamma \vec{\nu} + r \sin \gamma \vec{\nu}^\perp) \vec{n} r dr d\gamma,$$

which proves the result.

This theorem allows for the reconstruction of $\mu(x, y, z)$ cone beam measurements. It suggests that the cone beam measurements $C_\mu(s, \vec{m})$ be organized by integrating then along all integration planes passing through the source $S$, by applying Equation(94), i.e. for all $\vec{n}(\alpha, \beta)$. The result is a three-dimensional set of mesurements, $X(s, \alpha, \beta)$ corresponding to $s$ along the curve $\Gamma(s)$, $\alpha$ and $\beta$. Second, differentiating this scalar function $X(s, \alpha, \beta)$ with respect to $\beta$ gives $\frac{\partial X}{\partial \beta}$ which the theorem equates to $\frac{\partial R_\mu}{\partial \beta}(d, \alpha, \beta)$, where $d = \overrightarrow{OS} \vec{n}$. Finally, the 3D Radon Inverse Formula states

$$\mu(x, y, z) = -\frac{1}{8\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial^2 R_\mu}{\partial d^2}(x \cos \alpha \sin \beta + y \sin \alpha \sin \beta + z \cos \beta, \alpha, \beta) d\alpha d\beta$$
where
\[
\frac{\partial}{\partial d} \left( \frac{\partial R_{\mu}}{\partial d} \right) = \frac{\partial}{\partial d} \left( \frac{\partial X}{\partial \beta} \right) = \frac{\partial^2 X}{\partial s \partial \beta} = \frac{\partial^2 X}{\partial (\Gamma(s), \vec{n})} = \frac{\partial^2 X}{\partial s \partial \beta} \Gamma'(s). \vec{n}.
\]

6.3 Reconstruction Algorithms for Cone Beam Helical CT

Discuss why helix: Tuy’s completeness condition; long object problem

The helix is the most practical and popular source trajectory. Let \( \Gamma \) be the helix with radius \( R \) and pitch \( h \), i.e.,
\[
\Gamma(s) = (R \cos s, R \sin s, \frac{h}{2\pi} s).
\]

The Tam-Danielsson Window [27, 4] for each point of a helix is the region on the detector plane bounded by the projections of the upper and lower turns of a helical trajectory, Figure 24(a). Tam et al. [27] using the triangular partition procedure showed that exact reconstruction based on the 3D Radon Transform only requires cone beam data in the Tam-Danielsson windows. Then, for any region of interest only the Tam-Danielsson windows corresponding to this region are required.

A \( \pi \)-line is a straight line connecting two points of a helix which are less than a full \( 2\pi \) turn apart [4], Figure 24(b). The reason for the \( \pi \)-line terminology is that the data corresponding to the source locus on the helix segment between the two ends of each \( \pi \)-line covers \( 180^\circ \) of data for any point on the \( \pi \)-line [4] conjectured that the cone-beam data on this helix segment (axially truncated) was sufficient to reconstruct the object on the \( \pi \)-line and this was first shown by Katsevich [17, 14]. The cylinder containing the helix is called the helix cylinder. A \( \pi \)-line segment is the portion of the \( \pi \)-line enclosed by the helix cylinder. The \( \pi \)-helix segment on the \( \pi \)-arc is the segment of the helix bounded by the two end points of a \( \pi \)-line segment. The support cylinder of is the smallest cylinder radius \( R_0 < R \) through the same central axis containing the object of interest.
Observe that the tangent for the helix is
\[
\vec{\tau} = \frac{\Gamma'(s)}{|\Gamma''(s)|} = \left( -R \sin s, R \cos s, \frac{h}{2\pi} \right) = \left( -\sin s, \cos s, \frac{h}{2\pi R} \right) \sqrt{1 + \frac{h^2}{4\pi^2 R^2}}
\]
while the normal is
\[
\vec{n} = \frac{d\vec{\tau}}{ds} = (-\cos s, -\sin s, 0).
\]
We see that the normal to the helix is orthogonal to and goes through its central axis.

The following proposition states that any region of interest in a helical cylinder can completely be filled by \(\pi\)-line segments if there is data for a sufficient number of turns of the helix.

**Theorem 7.** [4, 5] Every point \(P(x, y, z) = P(r \cos \phi, r \sin \phi, z)\) in a helix cylinder belongs to one and only one \(\pi\)-line segment, bounded by \(s_1(P)\) and \(s_2(P)\), as given by solving Equations 137.

Next, to arrive at Katsevich’s exact reconstruction formula based on a filtered backprojection algorithm [14], we first need to define the notion of a \(K\)-plane which is a plane that intersects the helix at three equi-spaced points, i.e. at
Γ(s − Δs), Γ(s), and Γ(s + Δs) for some s and Δs, and denoted as K(s, Δs). It is clear that the K-plans associated to any source point Γ(s) form a one-parameter family indexed by Δs. In the limit, as Δs → 0, the K-plane is the bi-osculating plane (Ben to check the terminology) of the curve Γ, namely, the plane going through both the tangent and the normal to Γ, i.e., the plane orthogonal to the helical axis.

Figure 25: For each point P and any source point Γ(s) with \( s \in [s_1(p), s_2(p)] \), there exists a K-plane, K(s, Δs) passing through Γ(s) and P. The normal to this K-plane is \( \vec{m} \).

**Proposition 7.** The K-plane K(s, Δs) going through any point P(x, y, z) = P(r cos φ, r sin φ, z), and source Γ(s) on a helix is obtained from Δs satisfying

\[
sinc(\Delta s) = \frac{-hr \sin(s - \phi)}{2\pi R \ z - \frac{h}{R} s}.
\]

Ben: Existence of solution needs work.

**Proof.** The K-plane passing through Γ−1 = Γ(s − Δs), Γ0 = Γ(s), Γ1 = Γ(s + Δs) can be characterized as the plane through Γ0 with normal along \( \overrightarrow{\Gamma_0 \Gamma_1} \). The point P(x, y, z) is on this plane if and only if

\[
\overrightarrow{\Gamma_0 P} \cdot (\overrightarrow{\Gamma_1} \overrightarrow{\Gamma_0} \times \overrightarrow{\Gamma_0 \Gamma_1}) = 0.
\]
Since $\Gamma_0$ (or $s$) and $P$ are both fixed, only $\Delta s$ is unknown leading to one equation and one unknown. Using the equation for the helix,

\[
\begin{align*}
\Gamma_{-1} &= \Gamma(s - \Delta s) = \left( R \cos(s - \Delta s), R \sin(s - \Delta s), \frac{h}{2\pi} (s - \Delta s) \right) \\
\Gamma_0 &= \Gamma(s) = \left( R \cos s, R \sin s, \frac{h}{2\pi} \right) \\
\Gamma_1 &= \Gamma(s + \Delta s) = \left( R \cos(s + \Delta s), R \sin(s + \Delta s), \frac{h}{2\pi} (s + \Delta s) \right),
\end{align*}
\]

which gives

\[
\overrightarrow{\Gamma_0 \Gamma_1} = \left( R \cos(s + \Delta s) - R \cos s, R \sin(s + \Delta s) - R \sin s, \frac{h}{2\pi} \Delta s \right)
= \left( -2R \sin \left( \frac{\Delta s}{2} \right) \sin \left( s + \frac{\Delta s}{2} \right), 2R \sin \left( \frac{\Delta s}{2} \right) \cos \left( s + \frac{\Delta s}{2} \right), \frac{h}{2\pi} \Delta s \right),
\]

and similarly

\[
\overrightarrow{\Gamma_0 \Gamma_{-1}} = \left( 2R \sin \left( \frac{\Delta s}{2} \right) \sin \left( s - \frac{\Delta s}{2} \right), -2R \sin \left( \frac{\Delta s}{2} \right) \cos \left( s - \frac{\Delta s}{2} \right), \frac{-h}{2\pi} \Delta s \right).
\]

Thus,

\[
\overrightarrow{\Gamma_0 \Gamma_1} \times \overrightarrow{\Gamma_0 \Gamma_{-1}} = \left( \frac{Rh}{\pi} \Delta s \sin \left( \frac{\Delta s}{2} \right) \left[ -\cos \left( s + \frac{\Delta s}{2} \right) + \cos \left( s - \frac{\Delta s}{2} \right) \right],
\right.
\]

\[
\left. \frac{-Rh}{\pi} \Delta s \sin \left( \frac{\Delta s}{2} \right) \left[ \sin \left( s - \frac{\Delta s}{2} \right) - \sin \left( s + \frac{\Delta s}{2} \right) \right],
\right.
\]

\[
4R^2 \sin^2 \left( \frac{\Delta s}{2} \right) \left[ \sin \left( s + \frac{\Delta s}{2} \right) \cos \left( s - \frac{\Delta s}{2} \right) - \cos \left( s + \frac{\Delta s}{2} \right) \sin \left( s - \frac{\Delta s}{2} \right) \right],
\]

\[
= \left( \frac{Rh}{\pi} \Delta s \sin \left( \frac{\Delta s}{2} \right) \cdot 2 \sin \left( \frac{\Delta s}{2} \right) \sin s,
\right.
\]

\[
\left. \frac{-Rh}{\pi} \Delta s \sin \left( \frac{\Delta s}{2} \right) \cdot 2 \sin \left( \frac{\Delta s}{2} \right) \cos s,
\right.
\]

\[
4R^2 \sin^2 \left( \frac{\Delta s}{2} \right) \sin \Delta s
\]

\[
= 2R \sin^2 \left( \frac{\Delta s}{2} \right) \left( \frac{h}{\pi} \Delta s \sin s, \frac{-h}{\pi} \Delta s \cos s, 2R \sin \Delta s \right).
\]

Now,

\[
\overrightarrow{\Gamma_0 P} = \left( x - R \cos s, y - R \sin s, z - \frac{h}{2\pi} s \right),
\]

(98)
so that
\[ \overrightarrow{\Gamma_0 P} \cdot (\overrightarrow{\Gamma_0 \Gamma_1} \times \overrightarrow{\Gamma_0 \Gamma_{-1}}) = 2R \sin^2 \left( \frac{\Delta s}{2} \right) \left[ \frac{h}{\pi} \Delta s \sin s(x - R \cos s) + \frac{-h}{\pi} \Delta s \cos s(y - R \sin s) + 2R \sin \Delta s \left( z - \frac{h}{2\pi} s \right) \right] \]

Setting this equal to zero gives (Ben: What about \( \Delta s = 0? \))

\[ \text{sinc}(\Delta s) = \frac{\sin \Delta s}{\Delta s} = -\frac{h(x \sin s - y \cos s)}{2\pi R \left( z - \frac{h}{2\pi} s \right)}. \]

Using the cylindrical coordinates of \( P \) simplifies the equation to

\[ \text{sinc}(\Delta s) = -\frac{hr \sin(s - \phi)}{2\pi R \left( z - \frac{h}{2\pi} s \right)}. \] (99)

from which \( \Delta s \) can be obtained. Note that the solution to this equation may not exist, may be unique or multi-valued.

**Corollary 7.** The normals defined by the one-parameter family of \( K \)-planes \( K(s, \Delta s) \) through a given point \( \Gamma(s) \) forms a curve on the 2D unit sphere of all possible directions given by

\[ \vec{m}(s, \Delta s) = \frac{\left( \sin s, \cos s, \frac{-2\pi R \text{sinc}(\Delta s)}{h} \right)}{\sqrt{1 + \frac{4\pi^2 R^2}{h^2} \text{sinc}^2(\Delta s)}}. \] (100)

Thus, for a given point \( P(x, y, z) \) and source \( \Gamma_1 = \Gamma(s) \) we may not have a \( K \)-plane. However, varying \( s \) may lead to a solution.

**Remark 1.** It is clear from Equation 96 that the \( K \)-plane with respect to source \( \Gamma_0 \) of a point \( \hat{P} \) lying on the \( K \)-plane of \( P \) and source \( \Gamma_0 \) is the same \( K \)-plane specifically, the \( K \)-plane for all points along a given unit direction \( \vec{\Theta} \) passing through \( \Gamma_0 \), i.e., \( P(\tau) = \Gamma_0 + \tau \vec{\Theta} \) is the same as long as for \( \Gamma_0 = \Gamma(s), s \in [s_1(P), s_2(P)] \). Thus, a unit direction \( \vec{\Theta} \) defines the \( K \)-planes though \( \Gamma_0 = \Gamma(s) \). We let \( m(s, \vec{\Theta}) \) define the normal to such a \( K \)-plane.

For any point \( P \) and any source point \( \Gamma(s), s \in [s_1(P), s_2(P)] \), where \( s_1(P) \) and \( s_2(P) \) are source parameters corresponding to the end points of \( \pi \)-line of...
there exists a $K$-plane $K(s, \Delta s)$ passing through $\Gamma(s)$ and $P$ solve through Equation 99. (Ben: This still needs work.) Denote the normal to this $K$-plane as $\vec{m}(s, \tilde{\Theta})$, where $\tilde{\Theta}$ is the unit direction from $\Gamma(s)$ to $P$. This notation is permissible since we showed in Remark 1 that the $K$-plane is only dependent on $\tilde{\Theta}$ not on $P$. Thus, the significance of a $K$-plane is in defining a local direction $\vec{m}(s, \tilde{\Theta})$, for any given direction $\tilde{\Theta}$, Figure 25. Note that this defines a local coordinate system $(\tilde{\Theta}, \vec{m}, \vec{m}^\perp)$ where $\vec{m}^\perp = \tilde{\Theta} \times \vec{m}$.

The strategy underlying Katsevich’s formulation is to consider $K$-planes through a source point $\Gamma_0 = \Gamma(s)$ defined by any given direction $\tilde{\Theta}$. For each $K$-plane, a kernel $C^\ast_{\mu}(s, \tilde{\Theta})$ is defined and the object is reconstructed from it.

**Theorem 8** (Katsevich’s inversion formula [14]). The object $\mu(x, y, z)$ can be reconstructed as

$$\mu(P) = -\frac{1}{2\pi^2} \int_{s_1(P)}^{s_2(P)} C^\ast_{\mu} \left( s, \frac{P - \Gamma(s)}{|P - \Gamma(s)|} \right) \frac{1}{|P - \Gamma(s)|} ds. \quad (101)$$

where

$$C^\ast_{\mu}(s, \tilde{\Theta}) = \frac{1}{\pi} \int_0^{2\pi} \frac{\partial C_{\mu}}{\partial \tilde{s}} \left( \tilde{s}, \cos \gamma \tilde{\Theta} + \sin \gamma \vec{m}^\perp(s, \tilde{\Theta}) \right) \left| \frac{d}{ds} \tilde{s} \right| \frac{1}{\sin \gamma} d\gamma. \quad (102)$$

Note that the differentiation is with respect to the first occurrence of $s$ while leaving the one affecting $m^\perp$ constant. Then, Katsevich showed that the convolution nature of the computation defining $C^\ast_{\mu}(s, \tilde{\Theta})$ in Equation 102 is not readily apparent. This aspect is revealed when we fix the $K$-plane $K(s, \Delta s)$ through the source point $\Gamma(s)$, say with normal $m_0(s, \Delta s)$, and consider all $\tilde{\Theta}$ in this plane. Consider an arbitrary orthonormal coordinate frame for the $K$-plane $(\vec{e}_1, \vec{e}_2)$. Then, any unit vector $\tilde{\Theta}$ in this plane can be expressed in parametrized form as

$$\tilde{\Theta}(\tilde{s}) = \cos \tilde{s} \vec{e}_1 + \sin \tilde{s} \vec{e}_2. \quad (103)$$

This gives

$$\vec{m}^\perp(s, \tilde{\Theta}) = \tilde{\Theta} \times \vec{m}_0 \quad (104)$$

$$= (\cos \tilde{e}_1 + \sin \tilde{e}_2) \times \vec{m}_0 \quad (105)$$

$$= -\cos \tilde{e}_2 + \sin \tilde{e}_1 \quad (106)$$

$$= \sin \tilde{e}_1 - \cos \tilde{e}_2. \quad (106)$$
Then

\[ C_\mu^*(s, \tilde{\Theta} (\tilde{\gamma})) = \frac{1}{\pi} \int_0^{2\pi} \frac{\partial C_\mu}{\partial s} (s, \cos \gamma \tilde{e}_1 + \sin \gamma \tilde{e}_2) \mid_{\tilde{s}=s} \frac{1}{\sin \gamma} d\gamma \]

\[ = \frac{1}{\pi} \int_0^{2\pi} \frac{\partial C_\mu}{\partial s} (s, \cos \gamma \cos \tilde{\gamma} \tilde{e}_1 + \cos \gamma \sin \tilde{\gamma} \tilde{e}_2 + \sin \gamma \sin \tilde{\gamma} \tilde{e}_1 - \sin \gamma \cos \tilde{\gamma} \tilde{e}_2) \mid_{\tilde{s}=s} \frac{1}{\sin \gamma} d\gamma \]

\[ = \frac{1}{\pi} \int_0^{2\pi} \frac{\partial C_\mu}{\partial s} (s, \cos(\tilde{\gamma} - \gamma) \tilde{e}_1 + \sin(\tilde{\gamma} - \gamma) \tilde{e}_2) \frac{1}{\sin \gamma} d\gamma \]

\[ = \frac{1}{\pi} \frac{\partial C_\mu}{\partial s} (s, \cos \gamma \tilde{e}_1 + \sin \gamma \tilde{e}_2) *_\gamma \frac{1}{\sin \gamma}. \]

This suggests an algorithm: For each source point \( \Gamma(s) \) and \( K \)-plane \( K(s, \Delta s) \) with unit vector \( m_0 \), define an arbitrary orthonormal coordinate frame \( (\tilde{e}_1, \tilde{e}_2) \) in the plane and define

\[ F(s, \Delta s, \gamma) = \frac{\partial C_\mu}{\partial s} (s, \cos \gamma \tilde{e}_1 + \sin \gamma \tilde{e}_2). \]

Then obtain

\[ F^*(s, \Delta s, \gamma) = F(s, \Delta s, \gamma) *_\gamma \frac{1}{\sin \gamma}. \]

(To be completed.)

(Ben: Work out circular and flat panel detector algorithms @ [21])

6.4 History of Helical Cone Beam CT Development

[26] and [19] are extensions of Grangeat’s theory [8].

Exact algorithms have been based on Katsevich’s formula [15], which was a rather important breakthrough, in switching from Radon Transform-based methods to a filtered backprojection method with 1D shift-invariant filtering. Detector-plane parametrization methods [21] from [38] based on the Radon Transform require more (over-scan) data than those based on Katsevich’s formula in accurate reconstruction of long objects. On the other hand, Katsevich algorithms require more data with the Tam-Danielsson Window.

Zou and Pan [38] derived an exact formula for helical cone-beam reconstruction and developed a backprojection filtering algorithm where the cone-beams are first backprojected to \( \pi \)-line segments and then a 1D shift-invariant filtering is done over the \( \pi \)-segments. In [39], a filtered-backprojection algorithm
is developed where a 1D shift-invariant filtering is performed on the detector plane and then reconstruc the data by backprojecting. This is done by re-expressing the derivation in Katsevich’s formula [15, 17], Equations 96 and 97, in terms of coordinates on the detector plane. Since the number of samples along the helix are generally fewer than the dense samples along the detector plane, this is claimed to be numerically superior [37]. Katsevich [14] derived a formula with no derivatives that contains five distinct terms. Zou and Pan [35] developed an algorithm with three terms, but which contain partial derivatives with respect to detector plane coordinates.

Generalized Feldkamp-type algorithms were developed for the long object reconstruction with longitudinally truncated data obtained in a helical cone beam acquisition [31, 32].

Chen [2] gave an alternative proof of the Katsevich algorithm using Tuy’s formula [29]. Zhao et al. [34] extended this proof to include the Zou-Pan algorithm [38, 36] which is not only valid for helical scan but is generally valid for various nonstandard spirals and other curves; see also [33].


Quasi-exact: Defrise etal 2000, Kudo etal 2000, Tam 2000, zero boundary method, some data is processed wrong, Tam 2002


6.5 Material to be worked into this section

(Ben, it was not noted where to add the third hand-written page of notes, so I added it here. Please mark its correct location.) (Ben to work on this) Consider all planes through a point $P$ and a source point $\Gamma_0 = \Gamma(s)$ with normal $\vec{m}$. Then, there are three types of planes intersecting the $\pi$ helix segment at one, two, or three points [15]. The planes with two intersections do not contribute to the reconstructed image!
Corollary 8. The image $\mu$ at the point $P(x, y, z)$ can be fully reconstructed inside the helix cylinder by computing $\mu$ along the $\pi$-line segment containing $P$, i.e., the line segment bounded by $s_1(P)$ and $s_2(P)$ as given by Proposition 11.

Theorem 9. [38]
\[
\int \int \int K(x, y, z, \bar{x}, \bar{y}, \bar{z}) g(\bar{x}, \bar{y}, \bar{z}) \, dx \, dy \, dz,
\]
where
\[
K(x, y, z, \bar{x}, \bar{y}, \bar{z}) = \frac{1}{2\pi^2} \int \int \int \text{sgn}(\nabla \cdot \mathbf{e}_\pi(x, y, z)) e^{2\pi i \cdot (\bar{x}-x, \bar{y}-y, \bar{z}-z)} d\nabla,
\]
and where $e_\pi(x, y, z)$ is defined based on the unique $\pi$-line passing through $(x, y, z)$ which is anchored at $\Gamma_1 = \Gamma(s_1)$ and $\Gamma_2 = \Gamma(s_2)$
\[
e_\pi(x, y, z) = \frac{\Gamma_2 - \Gamma_1}{|\Gamma_2 - \Gamma_1|};
\]

Theorem 10. [17]
\[
\mu(x, y, z) = -\frac{1}{2\pi^2} \int_{s_1(x, y, z)}^{s_2(x, y, z)} \int_0^{2\pi} \frac{\partial}{\partial \theta} C_\mu(\Gamma(q), \Theta(s, x, y, z, \gamma)) |_{q=s} \, d\theta \, |(x, y, z) - \Gamma(s)| \sin \gamma
\]
where
\[
\Theta(s, x, y, z, \gamma) = \cos \gamma \bar{n}(s, x, y, z) + \sin \gamma \bar{e}(s, x, y, z)
\]
and
\[
e(s, x, y, z) = \bar{n}(s, x, y, z) \times u(s, x, y, z).
\]

Ben to work on unifying the notation; $u$ is not explained.

6.6 Numerical algorithms for helical cone-beam reconstruction

Kohler et al. [18] compared two approximate algorithms PI-SLANT [28] and WEDGE [30] with the exact method of Katsevich [15], Figures 27, 28 and 29. They showed that a modified WEDGE-PI method performs almost as well as the Katsevich method for a 64-row detector while PI-SLANT shows artifacts. For a 128-row detector, some artifacts also appear in the WEDGE-PI results, but the Katsevich method is about an order of magnitude slower, Figure 30.
\[
\begin{align*}
  f(x) &= -\frac{1}{2\pi^2} \left[ \frac{1}{\|y - x\|^2} \int_0^{2\pi} D_\gamma(y(s), \Theta(x, x, y)) \frac{dy}{\sin \gamma} \right] \cdot \left[ \frac{1}{\|y - x\|^2} \int_0^{2\pi} D_\gamma(y(s), \Theta(x, x, y)) \frac{dy}{\sin \gamma} \right] \cdot \left[ \frac{1}{\|y - x\|^2} \int_0^{2\pi} D_\gamma(y(s), \Theta(x, x, y)) \frac{dy}{\sin \gamma} \right] \\
  &= -\int_{\mathbb{S}^1(1)} \left( \frac{\partial}{\partial s} \frac{1}{\|y - x\|^2} \right) \int_0^{2\pi} D_\gamma(y(s), \Theta(x, x, y)) \frac{dy}{\sin \gamma} ds \\
  &= -\int_{\mathbb{S}^1(1)} \frac{\partial}{\partial s} \left( \frac{\partial}{\partial s} \frac{1}{\|y - x\|^2} \right) \int_0^{2\pi} (\nabla_{y(s)} D_\gamma)(y(s), \Theta(x, x, y)) \cdot dy ds \\
  &= -\int_{\mathbb{S}^1(1)} \frac{\partial}{\partial s} \left( \frac{\partial}{\partial s} \frac{1}{\|y - x\|^2} \right) \int_0^{2\pi} (\nabla_{y(s)} D_\gamma)(y(s), \Theta(x, x, y)) \cdot dy ds \\
  &= -\int_{\mathbb{S}^1(1)} \frac{\partial}{\partial s} \left( \frac{\partial}{\partial s} \frac{1}{\|y - x\|^2} \right) \int_0^{2\pi} \left( \frac{\partial}{\partial \gamma} \right) (\nabla_{y(s)} D_\gamma)(y(s), \Theta(x, x, y)) \frac{dy}{\sin \gamma} ds.
\end{align*}
\]

Figure 26: Equation taken from [17]

7 Radon’s Proof
Figure 27: Illustration of the PI-SLANT method. Left: Original PI acquisition; middle: Part of the focus-centered detector that is used; right: Geometry after parallel rebinning. Ramp filtering is performed along slanted lines on the virtual detector.

8 To Do

- Re-examine Radon’s formula in the light of the discontinuity in $\rho_\xi$. See [13] page 553, equation 3 and resulting $F_\xi(t)$.
- X-ray lenses.
- Summarize ART methods. [6, 12]
- Can to get refs 2,5,7,9,10,15 from this paper, and to check INRIA report corresponding to the TMI paper.
- Put in some description of flat panel x-ray detectors, x-ray tubes
Figure 28: Illustration of the WEDGE method. Left: Acquisition with a focus-centered detector; middle: Wedge geometry obtained by parallel rebinning. Ramp filtering is performed along rows of this virtual detector; right: In WEDGE-PI, the filtered data are height rebinned to the PI geometry to ensure that during back-projection, each object point gets contribution over exactly 180°.

Figure 29: Illustration of the KATSEVICH method. Left: Acquisition with a focus-centered detector; middle: Convolution with $\frac{1}{\sin \theta}$ along $K$ lines defined on a planar detector; right: Parallel rebinning of the filtered data to allow for back-projection without density correction.
Figure 30: Results of the 128 row scanner. From top to bottom: PI-SLANT, WEDGE-PI, KATSEVICH.

Figure 31:
A Integrals Involving Gaussians and Derivatives of Gaussians

A.1 Integrals involving a Gaussian and its derivatives

In the formulations below, all constants (upper case letters) are positive unless otherwise stated.

1. Integrals of Gaussians

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}, \quad \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} \, dx = \sqrt{2\pi\sigma}. \] (107)

\[ \int_{-\infty}^{\infty} e^{-Ax^2} \, dx = \frac{1}{\sqrt{A}} \int_{-\infty}^{\infty} e^{-y^2} \, dy = \sqrt{\frac{\pi}{A}}. \] (108)

\[ \int_{-\infty}^{\infty} e^{-(Ax^2+Bx+C)} \, dx = \int_{-\infty}^{\infty} e^{-A(x^2+\frac{B}{A}x+\frac{C}{A})} \, dx = \int_{-\infty}^{\infty} e^{-A[(x+\frac{B}{2A})^2+(\frac{C-\frac{B^2}{4A}}{A})]} \, dx \\
= e^{-A(\frac{C}{A} - \frac{B^2}{4A^2})} \int_{-\infty}^{\infty} e^{-A(x+\frac{B}{2A})^2} \, dx = e^{(\frac{B^2}{4A}-C)} \sqrt{\frac{\pi}{A}}. \] (109)

2. Integrals of Gaussians times \(x\)

\[ \int_{-\infty}^{\infty} xe^{-Ax^2} \, dx = 0. \]

\[ \int_{-\infty}^{\infty} xe^{-A(x+D)^2} \, dx = \int_{-\infty}^{\infty} (y - D)e^{-Ay^2} \, dy = -D \sqrt{\frac{\pi}{A}}. \]

\[ \int_{-\infty}^{\infty} xe^{-(Ax^2+Bx+C)} \, dx = e^{(\frac{B^2}{4A} - C)} \int_{-\infty}^{\infty} xe^{-A(x+\frac{B}{2A})^2} \, dx = e^{(\frac{B^2}{4A} - C)} \left( \frac{-B}{2A} \right) \sqrt{\frac{\pi}{A}}. \]

3. Integrals of Gaussians times \(x^2\)

\[ \int_{-\infty}^{\infty} x^2 e^{-x^2} \, dx = \left[ -\frac{x}{2} e^{-x^2} \right]_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}. \]

\[ \int_{-\infty}^{\infty} x^2 e^{-Ax^2} \, dx = \left[ -\frac{x}{2A} e^{-Ax^2} \right]_{-\infty}^{\infty} + \frac{1}{2A} \int_{-\infty}^{\infty} e^{-Ax^2} \, dx = \frac{1}{2A} \sqrt{\frac{\pi}{A}}. \]
\[ \int_{-\infty}^{\infty} x^2 e^{-(Ax^2+Bx+C)} \, dx \quad = \quad e^{\frac{B^2}{4A} - C} \int_{-\infty}^{\infty} x^2 e^{-\left(\frac{Bx}{A}\right)^2} \, dx = e^{\frac{B^2}{4A} - C} \int_{-\infty}^{\infty} \left( y - \frac{B}{2A} \right)^2 e^{-Ay^2} \, dy \]

\[ = \quad e^{\frac{B^2}{4A} - C} \left[ \int_{-\infty}^{\infty} y^2 e^{-Ay^2} \, dy - \int_{-\infty}^{\infty} \frac{B}{A} ye^{-Ay^2} \, dy + \int_{-\infty}^{\infty} \frac{B^2}{4A^2} e^{-Ay^2} \, dy \right] \]

\[ = \quad e^{\frac{B^2}{4A} - C} \left( \frac{1}{2A} \sqrt{\frac{\pi}{A}} + \frac{B^2}{4A^2} \sqrt{\frac{\pi}{A}} \right). \]

### A.2 Fourier Transform of the Gaussian

**Fourier Transform of a Gaussian:** Let \( g(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}}. \) Then, its Fourier Transform is

\[ G(\omega) = \mathcal{F}\left\{ \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} \right\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} e^{-i\omega x} \, dx \]

\[ = \quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\left( \frac{x}{\sqrt{\sigma}} + \frac{ix\omega}{\sqrt{2\sigma^2}} \right)^2} e^{-\frac{x^2}{2\sigma^2}} \, dx \]

\[ = \quad e^{-\frac{\omega^2\sigma^2}{2}} \]

Note that \( \sigma \) is the spatial domain transforms to \( \frac{1}{\omega} \) in the spectral domain.

**Fourier Transform of a DOG (Difference of Gaussians):**

\[ \mathcal{F}\left\{ \frac{1}{\sqrt{2\pi\sigma_1}} e^{-\frac{x^2}{2\sigma_1^2}} - \frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{x^2}{2\sigma_2^2}} \right\} = e^{-\frac{\omega^2\sigma_1^2}{2} - e^{-\frac{\omega^2\sigma_2^2}{2}}} \]

The peak of this distribution can be found by \( \mathcal{F}'(\omega_0) = 0 \), giving

\[ \sigma_1^2 e^{-\frac{\omega_0^2}{2}} = \sigma_2^2 e^{-\frac{3\omega_0^2}{2}}, \]

which after simplification gives

\[ \omega_0^2 = 2 \ln \frac{\sigma_2^2}{\sigma_1^2} - \ln \frac{\sigma_2^2}{\sigma_1^2}. \]

If \( \sigma_2 = c\sigma_1 \) then

\[ \omega_0^2 = \left( \frac{2 \ln c^2}{c^2 - 1} \right) \frac{1}{\sigma_1^2}. \]

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A.3 Integrals of the Gabor Filter

Proposition 8. For $\alpha > 0$ and $\beta > 0$, we have

\[
\int_0^\infty e^{-\alpha x^2} \cos(\beta x) \, dx = \frac{1}{2} \sqrt{\pi} e^{-\beta^2 / 4\alpha}.
\]
\[
\int_0^\infty e^{-\alpha x^2} \sin(\beta x) \, dx = e^{-\beta^2 / 4\alpha} \int_0^\beta e^{y^2} \, dy = \frac{1}{2\alpha} \int_0^\beta e^{-\frac{\beta^2 - y^2}{4\alpha}} \, dy.
\]

Proof. The two integrals can be solved by computing

\[
\int_0^\infty e^{-\alpha x^2} e^{-i\beta x} \, dx = e^{-\frac{\beta^2}{4\alpha}} \int_0^\infty e^{-\alpha \left( x - \frac{i\beta}{2\alpha} \right)^2} \, dx.
\]

where we assume $\alpha, \beta > 0$. The above integral can be solved by a contour integral $\int_c e^{-\alpha(z - \frac{i\beta}{2\alpha})^2} \, dz$ along the closed contour $C$ defined in Figure 34. This contour integral is 0 because there are no poles inside the contour:

\[
0 = \int_c e^{-\alpha(z - \frac{i\beta}{2\alpha})^2} \, dz
\]

\[
= \int_0^R e^{-\alpha(x - \frac{i\beta}{2\alpha})^2} \, dx + \int_0^{\beta / 2\alpha} e^{-\alpha(R+iy - \frac{i\beta}{2\alpha})^2} \, idy + \int_0^0 e^{-\alpha x^2} \, dx + \int_0^{\beta / 2\alpha} e^{-\alpha(iy - \frac{i\beta}{2\alpha})^2} \, idy.
\]

Figure 34: Rectangle Integral Path in the complex plane, where the $x$-axis represents the real part and $y$-axis represents the imaginary part.

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Note that when $R \mapsto \infty$, \[ \int_0^\infty e^{-\alpha (x-i\frac{\beta}{\sqrt{\pi}})^2} dx = \int_0^\infty e^{-\alpha x^2} dx + i \int_0^\infty e^{-\alpha (iy-i\frac{\beta}{\sqrt{\pi}})^2} dy \]
\[= \frac{1}{2\sqrt{\pi/\alpha}} + i \int_{-\frac{\beta}{\sqrt{\pi}}}^0 e^{\alpha y^2} dy, \]
\[= \frac{1}{2\sqrt{\pi/\alpha}} - \frac{i}{\sqrt{\alpha}} \int_0^{\frac{\beta}{\sqrt{\pi}}} e^{\alpha y^2} dy. \]

\[\Box\]

**A.4**

**Proposition 9.**

\[\mathcal{F}^{-1}\{\omega |e^{-\frac{\omega^2}{2}}\} = \frac{1}{\pi \sigma^2} \left[1 - \frac{\xi}{\sigma^2} \int_0^\xi e^{-\frac{\omega^2}{2\sigma^2} + \xi^2} d\omega \right] = \frac{1}{\pi \sigma^2} \left[1 + \int_0^\xi \frac{d}{d\xi} e^{-\frac{\omega^2}{2\sigma^2} + \xi^2} d\omega \right] \]

(110)

**Proof.**

\[\mathcal{F}^{-1}\{\omega |e^{-\frac{\omega^2}{2}}\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega| e^{-\frac{\omega^2}{2}} e^{i \omega \xi} d\omega \]
\[= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega| e^{-\frac{\omega^2}{2}} \cos(\omega \xi) d\omega + i \int_{-\infty}^{\infty} |\omega| e^{-\frac{\omega^2}{2}} \sin(\omega \xi) d\omega \]
\[= \frac{1}{\pi} \int_{0}^{\infty} \omega e^{-\frac{\omega^2}{2}} \cos(\omega \xi) d\omega \]
\[= \left(\frac{-1}{\pi \sigma^2} e^{-\frac{\omega^2}{2}} \cos(\omega \xi)\right)_{0}^{\xi} - \frac{\xi}{\pi \sigma^2} \int_{0}^{\xi} e^{-\frac{\omega^2}{2}} \sin(\omega \xi) d\omega \]
\[= \frac{1}{\pi \sigma^2} \left(1 - \xi \int_{0}^{\infty} e^{-\frac{\omega^2}{2}} \sin(\omega \xi) d\omega \right). \]

(111)

Now, from Appendix(A.3),

\[\int_{0}^{\infty} e^{-\alpha \omega^2} \sin(\beta \omega) d\omega = \frac{1}{\sqrt{\alpha}} e^{-\frac{\alpha^2}{4\beta}} \int_{0}^{\frac{\beta}{\sqrt{\pi}}} e^{\xi^2} d\xi. \]

(112)

Using $\alpha = \frac{\sigma^2}{2}$ and $\beta = \xi$ in the above equation, we get

\[\int_{0}^{\infty} e^{-\frac{\omega^2}{2}} \sin(\omega \xi) d\omega = \frac{\sqrt{2}}{\sigma} e^{-\frac{\sigma^2}{4\xi^2}} \int_{0}^{\frac{\xi}{\sqrt{2\pi}}} e^{\xi^2} d\xi = \frac{1}{\sigma^2} \int_{0}^{\xi} e^{-\frac{(\sigma^2-\xi^2)}{2\sigma^2}} d\xi. \]

(113)

Substituting back in Equation(111), we have the result. \[\Box\]
Proposition 10.

\[ \mathcal{F}^{-1}\{i\omega|\omega e^{-\frac{\omega^2}{2\sigma^2}}\} = \frac{1}{\pi\sigma^2} \left[ -\frac{\xi}{\sigma^2} + \int_0^\xi \frac{d^2}{d\xi^2} e^{-\frac{(\xi^2-\zeta^2)}{2\sigma^2}} d\zeta \right] \]

\[ = -\frac{1}{\pi\sigma^4} \left[ \xi + \left( 1 - \frac{\xi^2}{2\sigma^2} \right) \int_0^\xi e^{-\frac{(\xi^2-\zeta^2)}{2\sigma^2}} d\zeta \right]. \quad (114) \]

Proof. Approach 1:

\[ \mathcal{F}^{-1}\{i\omega|\omega e^{-\frac{\omega^2}{2\sigma^2}}\} = \frac{1}{\pi\sigma^2} \frac{d}{d\xi} \mathcal{F}^{-1}\{|\omega|e^{-\frac{\omega^2}{2\sigma^2}}\} \]

\[ = \frac{1}{\pi\sigma^2} \frac{d}{d\xi} \left[ \int_0^\xi \frac{d}{d\xi} e^{-\frac{(\xi^2-\zeta^2)}{2\sigma^2}} d\zeta \right] \]

\[ = \frac{1}{\pi\sigma^2} \left[ -\frac{\xi}{\sigma^2} + \int_0^\xi \frac{d^2}{d\xi^2} e^{-\frac{(\xi^2-\zeta^2)}{2\sigma^2}} d\zeta \right]. \]

Approach 2 (Direct):

\[ \mathcal{F}^{-1}\{i\omega|\omega e^{-\frac{\omega^2}{2\sigma^2}}\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} i\omega|\omega e^{-\frac{\omega^2}{2\sigma^2}} e^{i\omega \xi} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} i\omega|\omega e^{-\frac{\omega^2}{2\sigma^2}} \cos(\omega \xi) d\xi - \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega|\omega e^{-\frac{\omega^2}{2\sigma^2}} \sin(\omega \xi) d\xi \]

\[ = \frac{1}{\pi} \left[ \int_0^\infty \omega^2 e^{-\frac{\omega^2}{2\sigma^2}} \sin(\omega \xi) d\xi \right] \]

\[ - \frac{\xi}{\sigma^2} \left[ \int_0^\infty e^{-\frac{\omega^2}{2\sigma^2}} \sin(\omega \xi) d\omega + \xi \int_0^\infty \omega e^{-\frac{\omega^2}{2\sigma^2}} \cos(\omega \xi) d\omega \right] \]

\[ = \frac{1}{\pi\sigma^2} \left[ \int_0^\infty e^{-\frac{\omega^2}{2\sigma^2}} \sin(\omega \xi) d\omega + \xi \int_0^\infty \omega e^{-\frac{\omega^2}{2\sigma^2}} \cos(\omega \xi) d\omega \right] \]

\[ + \frac{\xi}{\sigma^2} \left[ \int_0^\infty \omega e^{-\frac{\omega^2}{2\sigma^2}} \cos(\omega \xi) d\omega \right] \]

\[ - \frac{\xi}{\sigma^2} \left[ \int_0^\infty e^{-\frac{\omega^2}{2\sigma^2}} \sin(\omega \xi) d\omega - \frac{\xi}{\sigma^2} \int_0^\infty e^{-\frac{\omega^2}{2\sigma^2}} \sin(\omega \xi) d\omega \right] \]

\[ = \frac{1}{\pi\sigma^2} \left[ \left( 1 - \frac{\xi^2}{\sigma^2} \right) \int_0^\infty e^{-\frac{\omega^2}{2\sigma^2}} \sin(\omega \xi) d\omega + \frac{\xi}{\sigma^2} \right] \]

\[ = \frac{1}{\pi\sigma^2} \left[ \left( 1 - \frac{\xi^2}{\sigma^2} \right) \frac{1}{\sigma^2} \int_0^\xi e^{-\frac{(\xi^2-\zeta^2)}{2\sigma^2}} d\zeta + \frac{\xi}{\sigma^2} \right] \]

\[ = \frac{1}{\pi\sigma^2} \left[ -\frac{\xi}{\sigma^2} + \int_0^\xi \frac{d^2}{d\xi^2} e^{-\frac{(\xi^2-\zeta^2)}{2\sigma^2}} d\zeta \right]. \]
Approach 3:

\[
\mathcal{F}^{-1}\{i\omega |\omega|^2 e^{-\frac{\omega^2}{2\sigma^2}}\} = -\frac{1}{\pi} \int_0^\infty \omega^2 e^{-\frac{\omega^2}{2\sigma^2}} \sin(\omega \xi) d\xi
\]

\[
= -\frac{1}{\pi} \frac{d}{d\xi} \int_0^\infty \omega e^{-\frac{\omega^2}{2\sigma^2}} \cos(\omega \xi) d\xi
\]

\[
= \frac{1}{\pi} \frac{d}{d\xi} \int_0^\infty \omega e^{-\frac{\omega^2}{2\sigma^2}} \cos(\omega \xi) d\omega
\]

\[
= \frac{1}{\pi} \frac{d^2}{d\xi^2} \int_0^\infty e^{-\frac{\omega^2}{2\sigma^2}} \sin(\omega \xi) d\omega
\]

\[
= \frac{1}{\pi} \frac{d^2}{d\xi^2} \left[ \frac{1}{\sigma^2} \int_0^{\xi} e^{-\frac{(\xi - \zeta)^2}{2\sigma^2}} d\zeta \right]
\]

\[
= \frac{1}{\pi} \frac{d}{d\xi} \left[ 1 + \int_0^{\xi} \frac{d}{d\xi} \left( e^{-\frac{(\xi - \zeta)^2}{2\sigma^2}} \right) d\zeta \right]
\]

\[
= \frac{1}{\pi} \frac{d}{d\xi} \left[ -\frac{\xi}{\sigma^2} + \int_0^{\xi} \frac{d^3}{d\xi^3} \left( e^{-\frac{(\xi - \zeta)^2}{2\sigma^2}} \right) d\zeta \right].
\]

Proposition 11.

\[
\mathcal{F}^{-1}\{\omega^2 |\omega| e^{-\frac{\omega^2}{2\sigma^2}}\} = \frac{1}{\pi \sigma^2} \left[ \frac{2}{\sigma^2} - \frac{\xi^2}{\sigma^4} - \int_0^{\xi} \frac{d^3}{d\xi^3} e^{-\frac{(\xi - \zeta)^2}{2\sigma^2}} d\zeta \right].
\]

Proof.

\[
\mathcal{F}^{-1}\{\omega^2 |\omega| e^{-\frac{\omega^2}{2\sigma^2}}\} = -\frac{d}{d\xi} \mathcal{F}^{-1}\{i\omega |\omega| e^{-\frac{\omega^2}{2\sigma^2}}\}
\]

\[
= -\frac{1}{\pi \sigma^2} \frac{d}{d\xi} \left[ \frac{-\xi}{\sigma^2} + \int_0^{\xi} \frac{d^2}{d\xi^2} e^{-\frac{(\xi - \zeta)^2}{2\sigma^2}} d\zeta \right]
\]

\[
= -\frac{1}{\pi \sigma^2} \left[ \frac{-1}{\sigma^2} + \left( \frac{-1}{\sigma^2} + \frac{\xi^2}{\sigma^4} \right) + \int_0^{\xi} \frac{d^3}{d\xi^3} e^{-\frac{(\xi - \zeta)^2}{2\sigma^2}} d\zeta \right]
\]

\[
= \frac{1}{\pi \sigma^2} \left[ \frac{2}{\sigma^2} - \frac{\xi^2}{\sigma^4} - \int_0^{\xi} \frac{d^3}{d\xi^3} e^{-\frac{(\xi - \zeta)^2}{2\sigma^2}} d\zeta \right].
\]
B Integrals Involving Characteristic Function of Circular Domains

B.1

Proposition 12. The Fourier Transform of

\[ f(x) = \begin{cases} \sqrt{A^2 - x^2} & x \leq A \\ 0 & x > A. \end{cases} \]

is

\[ \mathcal{F}(\omega) = \frac{\pi A}{\omega} \left( \frac{\sin(\omega A)}{(\omega A)^2} - \frac{\cos(\omega A)}{\omega A} \right) \]

Proof.

\[ \mathcal{F}(\omega) = \int_{-A}^{A} \sqrt{A^2 - x^2} e^{-i\omega x} \, dx = 2 \int_{0}^{A} \sqrt{A^2 - x^2} \cos(\omega x) \, dx \quad (116) \]

It is well-known that \[7\] \[1\]

\[ \int_{0}^{u} (u^2 - x^2)^{v-\frac{1}{2}} \cos(ax) \, dx = \frac{\sqrt{\pi}}{2} \left( \frac{2u}{a} \right)^{v} \Gamma(v+\frac{1}{2}) J_{v}(au) \]

for \([a > 0, u > 0, Re(v) > -\frac{1}{2}], \) where \(\Gamma(z)\) is the gamma function

\[ \Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} \, dt, \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \]

and \(J_{v}(z)\) is the Bessel function of first kind,

\[ J_{v}(z) = \frac{z^{v}}{2^{v}} \sum_{k=0}^{\infty} (-1)^{k} \frac{z^{2k}}{2^{2k} k! \Gamma(v+k+1)}, \quad J_{1}(z) = \frac{z}{2} \sum_{k=0}^{\infty} (-1)^{k} \frac{z^{2k}}{2^{2k} k! \Gamma(k+2)} \]

for \(|\arg(z)| < \pi\), or in closed form

\[ J_{v}(z) = z^{v} \left( \frac{d}{dz} \right)^{v} \sin \frac{z}{z}, \quad J_{1}(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}. \]

Now, returning to Equation(116), we get

\[ 2 \int_{0}^{A} (A^2 - x^2)^{\frac{1}{2}} \cos(\omega x) \, dx = 2 \frac{\sqrt{\pi}}{2} \left( \frac{2A}{\omega} \right) \Gamma\left(\frac{3}{2}\right) J_{1}(\omega A) \quad \left[ \omega > 0, A > 0, Re(1) > -\frac{1}{2} \right] \]

\[ = \frac{\pi A}{\omega} J_{1}(\omega A) \]

\[ = \frac{\pi A}{\omega} \left( \frac{\sin(\omega A)}{(\omega A)^2} - \frac{\cos(\omega A)}{\omega A} \right). \]

\[ \square \]
B.2

Proposition 13.

\[ f(x) = \mathcal{F}^{-1}\{\left[|\omega|\frac{\pi R}{\omega} \left(\frac{\sin(\omega R)}{(\omega R)^2} - \frac{\cos(\omega R)}{\omega R}\right)\right]\} \]

\[ = 1 + \frac{x}{2R} \ln \left|\frac{x - R}{x + R}\right|, R > 0 \]

Proof. Using Maple8

\[ f(x) = -\frac{1}{4R} \left[-x\pi \text{csgn}((x + R)i)\text{csgn}(x + R)i + x\pi \text{csgn}((x - R)i)\text{csgn}(x - R)i + \right. \\
\[ x \ln(x + Ri) - x \ln(x - Ri) - 4R + x \ln(-xi - Ri) - x \ln(-xi + Ri)\left.\right]\]

where

\[ \text{csgn}(x) = \begin{cases} 
1 & \text{if } \text{Re}(x) > 0 \text{ or } (\text{Re}(x) = 0 \text{ and } \text{Im}(x) > 0) \\
-1 & \text{if } \text{Re}(x) < 0 \text{ or } (\text{Re}(x) = 0 \text{ and } \text{Im}(x) < 0) 
\end{cases} \]

First, when \( x < -R, x + R < 0 \text{ and } x - R < 0 \). Using \( \ln(i) = \frac{\pi i}{2} \), we have

\[ f(x) = -\frac{1}{4R} \left[-x\pi(1)(-1)i + x\pi(-1)(-1)i + x(\ln(-x - R) - \frac{\pi i}{2}) - x(\ln(R - x) - \frac{\pi i}{2})\right. \\
\[ -4R + x(\ln(-x - R) + \frac{\pi i}{2}) - x(\ln(R - x) + \frac{\pi i}{2})\left.\right]\]

\[ = 1 + \frac{x}{2R} \ln \left|\frac{R - x}{-(R + x)}\right| \]

Second, when \(-R < x < R\) we have \( x + R > 0 \) and \( x - R < 0 \), and

\[ f(x) = -\frac{1}{4R} \left[-x\pi(1)(1)i + x\pi(-1)(-1)i + x(\ln(x + R) + \frac{\pi i}{2}) - x(\ln(R - x) - \frac{\pi i}{2})\right. \\
\[ -4R + x(\ln(x + R) - \frac{\pi i}{2}) - x(\ln(R - x) + \frac{\pi i}{2})\left.\right]\]

\[ = 1 + \frac{x}{2R} \ln \left|\frac{R - x}{R + x}\right| \]

Finally, when \( R < x, x + R > 0 \) and \( x - R > 0 \), which gives

\[ f(x) = -\frac{1}{4R} \left[-x\pi(1)(1)i + x\pi(1)(1)i + x(\ln(x + R) + \frac{\pi i}{2}) - x(\ln(x - R) + \frac{\pi i}{2})\right. \\
\[ -4R + x(\ln(x + R) - \frac{\pi i}{2}) - x(\ln(R - x) - \frac{\pi i}{2})\left.\right]\]

\[ = 1 + \frac{x}{2R} \ln \left|\frac{x - R}{R + x}\right| \]
These cases can be combined to give the desired result.

\section{Some Standard Fourier Formulas}

In the following, $f(x)$ and $F(\omega)$ are considered as Fourier Transform pairs, i.e., $F(\omega) = \mathcal{F}[f(x)]$.

1. 
\[ \mathcal{F}\left\{\frac{d}{dx}f(x)\right\} = i\omega F(f(x)). \] (117)

2. Let 
\[ F(\omega) = \begin{cases} 
1, & |\omega| \leq \Omega \\
0, & |\omega| > \Omega
\end{cases} \] (118)

Then, 
\[ f(x) = \sin(\Omega x) \frac{\pi}{\Omega} \text{sinc}(\Omega x). \] (119)

3. Let 
\[ F(\omega) = \begin{cases} 
|\omega|, & |\omega| \leq \Omega \\
0, & |\omega| > \Omega
\end{cases} \] (120)

Then, 
\[ f(x) = \Omega^2 \frac{\pi}{\Omega} [\text{sinc}(\Omega x) - \frac{1}{2}\text{sinc}^2(\frac{\Omega x}{2})]. \] (121)

4. Let\footnote{This function is not one near omega 0 which is problematic.} 
\[ H_{BK}(\omega) = \begin{cases} 
\omega \sin\left(\frac{\pi}{\Omega} \omega\right), & |\omega| \leq \Omega \\
0, & |\omega| > \Omega
\end{cases} \] (122)
Then, the spatial domain Shepp-Logan filter is

\[ h_{BK}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_{BK}(\omega) e^{i\omega x} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \omega \sin \left( \frac{\pi}{2\Omega} \omega \right) e^{i\omega x} d\omega \]  

\[ = \frac{1}{\pi} \int_{0}^{\Omega} \omega \sin \left( \frac{\pi}{2\Omega} \omega \right) \cos(\omega x) d\omega \]

\[ = \frac{1}{\pi} \frac{d}{dx} \left[ \int_{0}^{\Omega} \sin \left( \frac{\pi}{2\Omega} \omega \right) \sin(\omega x) d\omega \right] \]

\[ = \frac{1}{2\pi} \frac{d}{dx} \left[ \int_{0}^{\Omega} \cos \left( \left( \frac{\pi}{2\Omega} - x \right) \omega \right) d\omega - \int_{0}^{\Omega} \cos \left( \left( \frac{\pi}{2\Omega} + x \right) \omega \right) d\omega \right] \]

\[ = \frac{1}{\pi} \frac{d}{dx} \cos(\Omega x) \left[ \frac{1}{\left( \frac{\pi}{2\Omega} - x \right)} - \frac{1}{\left( \frac{\pi}{2\Omega} + x \right)} \right] \]

\[ = \frac{-\Omega}{2\pi} \sin(\Omega x) \left[ \frac{1}{\left( \frac{\pi}{2\Omega} - x \right)} - \frac{1}{\left( \frac{\pi}{2\Omega} + x \right)} \right] + \frac{\cos(\Omega x)}{2\pi} \left[ \frac{1}{\left( \frac{\pi}{2\Omega} - x \right)^2} + \frac{1}{\left( \frac{\pi}{2\Omega} + x \right)^2} \right] \]

\[ = -\frac{\Omega}{\pi} \sin(\Omega x) \frac{\pi}{\frac{\pi^2}{4\Omega^2} - x^2} + \frac{\cos(\Omega x)}{\pi} \frac{\frac{\pi^2}{4\Omega^2} + x^2}{\left( \frac{\pi^2}{4\Omega^2} - x^2 \right)^2}. \]

\[ (123) \]

\[ (124) \]

\[ (125) \]

\[ (126) \]

\[ (127) \]

\[ (128) \]

\[ (129) \]

\[ (130) \]

\[ (131) \]

\[ (132) \]

\[ (133) \]

D A Numerically stable method to find an extremum

Consider the problem of finding the extremum of a function \( f(x) \), which by elementary calculus requires

\[ \begin{cases} f'(x_0) = 0 \\ f''(x_0) > 0 \end{cases} \] for a minimum, and \( \begin{cases} f'(x_0) = 0 \\ f''(x_0) < 0 \end{cases} \) for a maximum.
Since finding zeros of a function is not as stable as finding a zero-crossing, we write this in the equivalent form

\[
\begin{align*}
  f'(x_0 + \epsilon) > 0 & \quad \text{for a minimum, and} \\
  f'(x_0 - \epsilon) < 0 & \quad \text{for a maximum,}
\end{align*}
\]

where \( \epsilon \) is a sufficiently small number. This can be further expanded as

\[
\begin{align*}
  f(x_0 + \epsilon) > f(x_0) & \quad \text{for a minimum, and} \\
  f(x_0 - \epsilon) > f(x_0) & \quad \text{for a maximum.}
\end{align*}
\]

In summary form, let

\[
\rho_\epsilon(x_0) = \frac{1}{2} [\text{sgn}(f(x_0 + \epsilon) - f(x_0)) + \text{sgn}(f(x_0 - \epsilon) - f(x_0))].
\]

Then, in the limit as \( \epsilon \to 0 \), \( \rho_\epsilon(x_0) = 1 \) if and only if \( x_0 \) is a minimum and \( \rho_\epsilon(x_0) = -1 \) if and only if \( x_0 \) is a maximum.

E Solving \( A \sin \theta + B \cos \theta = C \)

Proposition 14. Solutions of \( A \sin \theta + B \cos \theta + C \) satisfy

\[
\sin \theta = \frac{AC \pm B \sqrt{A^2 + B^2 - C^2}}{A^2 + B^2}. \tag{134}
\]
Proof. Write $A \sin \theta + B \cos \theta + C$ as

$$(A \sin \theta - C)^2 = B^2 \cos^2 \theta = B^2(1 - \sin^2 \theta).$$

Thus

$$(A^2 + B^2) \sin^2 \theta - 2AC \sin \theta + (C^2 - B^2) = 0,$$

a quadratic equation in $\sin \theta$ giving solutions

$$\sin \theta = \frac{AC \pm \sqrt{A^2C^2 - (A^2 + B^2)(C^2 - B^2)}}{A^2 + B^2},$$

or

$$\sin \theta = \frac{AC \pm B\sqrt{A^2 + B^2 - C^2}}{A^2 + B^2}.$$  \hfill (135)

Proposition 15. Solution of

$A \sin \theta + B \cos \theta = C$

satisfy

$$\theta = \sin^{-1} \left( \frac{C}{\sqrt{A^2 + B^2}} - \alpha \right),$$

when $\alpha$ is defined as

$$(\cos \alpha, \sin \alpha) = \frac{(A, B)}{\sqrt{A^2 + B^2}}.$$  \hfill (136)

Proof. We have

$$\sqrt{A^2 + B^2} \cos \alpha \sin \theta + \sqrt{A^2 + B^2} \sin \theta \cos \theta = C,$$

which gives

$$\sin(\theta + \alpha) = \frac{C}{\sqrt{A^2 + B^2}},$$

and

$$\theta = \sin^{-1} \left( \frac{C}{\sqrt{A^2 + B^2}} \right) - \alpha.$$  \hfill □
\section*{F \textit{π}-line membership}

\textbf{Theorem 11.} \cite{4, 5} Every point \(P(x, y, z) = P(r \cos \phi, r \sin \phi, z)\) in a helix cylinder belongs to one and only one \(\pi\)-line segment, bounded by \(s_1(P)\) and \(s_2(P)\), as given by solving Equations 137.

\textit{Proof.} Consider a \(\pi\)-line segment bounded by \(\Gamma_1 = \Gamma(s_1)\) and \(\Gamma_2 = \Gamma(s_2)\) where by definition \(|s_2 - s_1| < 2\pi\). Points on this line segment are therefore of the form \(\overrightarrow{O \Gamma_1} + t \overrightarrow{O \Gamma_2}\), \(t \in [0, 1]\), which can be written as

\begin{equation}
(1 - t)\overrightarrow{O \Gamma_1} + t \overrightarrow{O \Gamma_2} = \left( R[(1 - t) \cos s_1 + t \cos s_2], R[(1 - t) \sin s_1 + t \sin s_2], \frac{h}{2\pi}[(1 - t)s_1 + ts_2]\right).
\end{equation}

Now, solving for \(P(x, y, z) = (1 - t)\overrightarrow{O \Gamma_1} + t \overrightarrow{O \Gamma_2}\), we have

\begin{equation}
\begin{cases}
\frac{x}{R} = (1 - t) \cos s_1 + t \cos s_2 \\
\frac{y}{R} = (1 - t) \sin s_1 + t \sin s_2 \\
\frac{2\pi z}{h} = (1 - t)s_1 + ts_2,
\end{cases} \tag{137}
\end{equation}

which is a system of three equations in three unknowns \(s_1, s_2\) and \(t\). The computation will be easier if we describe the point \(P(x, y, z)\) in cylindrical coordinates as \(P(x, y, z) = (r \cos \phi, r \sin \phi, z)\). Then

\begin{equation}
\begin{cases}
\frac{r}{R} \cos \phi = (1 - t) \cos s_1 + t \cos s_2 \\
\frac{r}{R} \sin \phi = (1 - t) \sin s_1 + t \sin s_2 \\
\frac{2\pi z}{h} = (1 - t)s_1 + ts_2,
\end{cases} \tag{138}
\end{equation}

The first two equations can be used to write \(s_1\) and \(t\) in terms of \(s_2\). The third equation can then be used to solve for \(s_2\). First, we can solve for \(t\) by squaring the \(\cos s_1\) and \(\sin s_1\) in each term and adding them up to one

\[
\left(\frac{r}{R} \cos \phi - t \cos s_2\right)^2 + \left(\frac{r}{R} \sin \phi - t \sin s_2\right)^2 = (1 - t)^2,
\]

or

\[
\frac{r^2}{R^2} + t^2 - 2t \frac{r}{R} \cos(\phi - s_2) = t^2 - 2t + 1,
\]

which gives

\begin{equation}
t(s_2) = \frac{1 - \frac{r^2}{R^2}}{2 \left(1 - \frac{r}{R} \cos(\phi - s_2)\right)}. \tag{139}
\end{equation}
Second, $s_1$ can be solved in terms of $s_2$ by substituting $t$ into the first of Equations 138. However, it is simpler to combine the first two equations directly.

$$\frac{r}{R}(\cos \phi \sin s_2 - \sin \phi \cos s_2) = (1 - t)(\cos s_1 \sin s_2 - \sin s_2 \cos s_2)$$

which gives

$$\frac{r}{R} \sin(s_2 - \phi) = (1 - t) \sin(s_2 - s_1),$$

so that

$$\sin(s_2 - s_1) = \frac{r}{R(1-t)} \sin(s_2 - \phi)$$

$$= \frac{\frac{r}{R}}{1 - \frac{1 - r^2}{2(1 - \frac{r}{R} \cos(\phi - s_2))}} \sin(s_2 - \phi)$$

$$= \frac{2r}{r^2 + 1 - 2r \cos(\phi - s_2)} \sin(s_2 - \phi)$$

$$= \frac{2r}{r^2 + R^2 - 2r R \cos(\phi - s_2)} \sin(s_2 - \phi).$$

(140)

i.e.

$$s_1(s_2) = s_2 - \sin^{-1} \left[ \frac{2r}{r^2 + R^2 - 2r R \cos(\phi - s_2)} \sin(s_2 - \phi) \right].$$

(141)

Third, the last equation in 138 can now be used to determine $s_2$ by substituting Equation 139 for $t$ and Equation 141 for $s_1$. However, this tends to a rather complex expression from which the existence and uniqueness of a solution is difficult to infer. Defrise, Noo and Kudo [5] followed a different approach. Consider the family of $\pi$-segments indexed by $s_2$ that go through the line parallel to the $z$-axis going through $P$, i.e., when $s_1$ and $t$ satisfy the first two equations of 138. In this case, $z$ is determined by $s_2$ via the third equation.

$$z(x, y, s_2) = \frac{h}{2\pi} \left[ (1 - t(x, y, s_2))s_1(x, y, s_2) + t(x, y, s_2)s_2 \right].$$

(142)

Observe that $\lim_{s_2 \to \pm \infty} z = \pm \infty$. Then if $\frac{\partial z}{\partial s_2}$ is positive, $z(x, y, s_2) = z_0$ has one and only one solution. We have

$$\frac{\partial z}{\partial s_2} = \frac{h}{2\pi} \left[ -\frac{\partial t}{\partial s_2} s_1 + (1 - t) \frac{\partial s_1}{\partial s_2} + t \right]$$

$$= \frac{h}{2\pi} \left[ t + (s_2 - s_1) \frac{\partial t}{\partial s_2} + (1 - t) \frac{\partial s_1}{\partial s_2} \right].$$

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Differentiating the first two equations in Equation 138 gives a linear system of equations in \( \frac{\partial t}{\partial s_2} \) and \( \frac{\partial s_1}{\partial s_2} \):

\[
\begin{cases}
- \frac{\partial t}{\partial s_2} \cos s_1 - (1 - t) \sin s_1 \frac{\partial s_1}{\partial s_2} + \frac{\partial t}{\partial s_2} \cos s_2 - t \sin s_2 = 0 \\
- \frac{\partial t}{\partial s_2} \sin s_1 + (1 - t) \cos s_1 \frac{\partial s_1}{\partial s_2} + \frac{\partial t}{\partial s_2} \sin s_2 + t \cos s_2 = 0.
\end{cases}
\]

This is simplified as

\[
\begin{cases}
(\cos s_2 - \cos s_1) \frac{\partial t}{\partial s_2} - (1 - t) \sin s_1 \frac{\partial s_1}{\partial s_2} = t \sin s_2 \\
(\sin s_2 - \sin s_1) \frac{\partial t}{\partial s_2} + (1 - t) \cos s_1 \frac{\partial s_1}{\partial s_2} = -t \cos s_2.
\end{cases} \tag{143}
\]

By eliminating \( \frac{\partial s_1}{\partial s_2} \), we get

\[
(\cos(s_2 - s_1) - 1) \frac{\partial t}{\partial s_2} = t \sin(s_2 - s_1),
\]

or

\[
\frac{\partial t}{\partial s_2} = \frac{t \sin(s_2 - s_1)}{(\cos(s_2 - s_1) - 1)}.
\]

Observe that

\[
\frac{\sin \theta}{\cos \theta - 1} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{1 - 2 \sin^2 \frac{\theta}{2} - 1} = \cot \frac{\theta}{2},
\]

which simplifies the expression for \( \frac{\partial t}{\partial s_2} \) as

\[
\frac{\partial t}{\partial s_2} = -t \cot \left( \frac{s_2 - s_1}{2} \right). \tag{144}
\]

Similarly, by eliminating \( \frac{\partial t}{\partial s_2} \) from 143

\[
(1 - t)(\cos(s_2 - s_1) - 1) \frac{\partial s_1}{\partial s_2} = -t + t \cos(s_2 - s_1)
\]

which gives

\[
\frac{\partial s_1}{\partial s_2} = \frac{t}{1 - t}. \tag{145}
\]

Thus,

\[
\frac{\partial z}{\partial s_2} = \frac{h}{2\pi} \left[ t - (s_2 - s_1)t \cot \left( \frac{s_2 - s_1}{2} \right) + (1 - t) \frac{t}{1 - t} \right]
\]

\[
= \frac{ht}{\pi} \left[ 1 - \frac{s_2 - s_1}{2} \cot \left( \frac{s_2 - s_1}{2} \right) \right]. \tag{146}
\]
Observe that the function $f(\theta) = 1 - \theta \cot \theta$ is always positive for $0 < \theta < \pi$ (Figure 36) so that for $0 < s_2 - s_1 < 2\pi$, the derivative $\frac{\partial z}{\partial s_2} > 0$. Since the function $z(x, y, s_2)$ is continuous and monotonic going from $(-\infty, \infty)$, the equation $z(x, y, s_2) = z_0$ has one and only one solution in $s_2$ and then in $(t(s_2), s_1(s_2), s_2)$ when Equations 139 and 141 are taken into account.

Figure 36: The plot of $1 - \theta \cot \theta$

Alternatively, we can eliminate $t$ directly from the first two equations of Equation 138 by first rewriting

\[
\begin{align*}
\frac{r}{R} \cos \phi &= \cos s_1 + t(\cos s_2 - \cos s_1) \\
\frac{r}{R} \sin \phi &= \sin s_1 + t(\sin s_2 - \sin s_1)
\end{align*}
\]

and then eliminating $t$

\[
\frac{r}{R} [\cos \phi (\sin s_2 - \sin s_1) - \sin \phi (\cos s_2 - \cos s_1)] = \cos s_1 (\sin s_2 - \sin s_1) - \sin s_1 (\cos s_2 - \cos s_1),
\]

which can be organized in two ways. First, we can write

\[
\frac{r}{R} [\sin(s_2 - \phi) - \sin(s_1 - \phi)] = \sin(s_2 - s_1),
\]

which is an implicit equation for solving $s_1$. Second, an explicit formula can be obtained by writing

\[
(\cos s_2 - \frac{r}{R} \cos \phi) \sin s_1 - (\sin s_2 - \frac{r}{R} \sin \phi) \cos s_1 = -\frac{r}{R} \sin(s_2 - \phi).
\]
This equation is the form of

\[ A \sin s_1 + B \cos s_1 = C \]

or can be solved as indicated in Appendix E, Equation 135, giving

\[ \sin s_1 = \frac{AC \pm B \sqrt{A^2 + B^2 - C^2}}{A^2 + B^2}. \]

In this case

\[ A^2 + B^2 = \left( \cos s_2 - \frac{r}{R} \cos \phi \right)^2 + \left( -\sin s_2 + \frac{r}{R} \sin \phi \right)^2 = 1 + \frac{r^2}{R^2} - \frac{2r}{R} \cos(s_2 - \phi), \]

and

\[ A^2 + B^2 - C^2 = 1 + \frac{r^2}{R^2} - \frac{2r}{R} \cos(s_2 - \phi) - \frac{r^2}{R^2} \sin^2(s_2 - \phi) \]

\[ = 1 + \frac{r^2}{R^2} \cos^2(s_2 - \phi) - 2 \frac{r}{R} \cos(s_2 - \phi) \]

\[ = \left[ 1 - \frac{r}{R} \cos(s_2 - \phi) \right]^2. \]

Thus,

\[ \sin s_1 = -\frac{r}{R} \left( \cos s_2 - \frac{r}{R} \cos \phi \right) \sin(s_2 - \phi) \mp \left( \sin s_2 - \frac{r}{R} \sin \phi \right) \left( 1 - \frac{r}{R} \cos(s_2 - \phi) \right) \]

\[ = \frac{\mp \sin s_2 \pm \frac{r}{R} \sin \phi - \left( \cos s_2 \sin(s_2 - \phi) + \frac{r^2}{R^2} \cos \phi \sin(s_2 - \phi) \mp \sin \phi \cos(s_2 - \phi) \right)}{1 + \frac{r^2}{R^2} - 2 \frac{r}{R} \cos(s_2 - \phi)}. \]

Taking the plus solution first

\[ \sin s_1 = \frac{-\sin s_2 + \frac{r}{R} \sin \phi - \frac{r}{R} \sin(-\phi) + \frac{r^2}{R^2} \sin(s_2 - 2\phi)}{1 + \frac{r^2}{R^2} - 2 \frac{r}{R} \cos(s_2 - \phi)} \]

\[ = \frac{r^2}{R^2} \sin(s_2 - 2\phi) - \sin s_2 + 2 \frac{r}{R} \sin \phi. \]

Taking the minus solution next

\[ \sin s_1 = \frac{\sin s_2 - \frac{r}{R} \sin \phi - \frac{r}{R} \sin(2s_2 - \phi) + \frac{r^2}{R^2} \sin s_2}{1 + \frac{r^2}{R^2} - 2 \frac{r}{R} \cos(s_2 - \phi)} \]

\[ = \frac{\left( \frac{r^2}{R^2} + 1 \right) \sin s_2 - \frac{r}{R} \sin(s_2 - \phi) - \frac{r}{R} \sin \phi}{1 + \frac{r^2}{R^2} - 2 \frac{r}{R} \cos(s_2 - \phi)}. \]
Alternatively, let
\[(\cos \alpha, \sin \alpha) = \left( \frac{r}{R} \cos \phi - \cos s_2, \frac{r}{R} \sin \phi - \sin s_2 \right) \sqrt{\frac{r^2}{R^2} + 1 - 2 \frac{r}{R} \cos(s_2 - \phi)} \] (147)

Then
\[\sin(s_1 - \alpha) = \frac{-\frac{r}{R} \sin(s_2 - \phi)}{\sqrt{1 + \frac{r^2}{R^2} - 2 \frac{r}{R} \cos(s_2 - \phi)}}\]
so that
\[s_1(s_2) = \alpha - \sin^{-1} \left[ \frac{-\frac{r}{R} \sin(s_2 - \phi)}{\sqrt{1 + \frac{r^2}{R^2} - 2 \frac{r}{R} \cos(s_2 - \phi)}} \right].\]

To be continued.

F.1 References not included in the paper yet

[16]

References


