Magnetic Resonance Imaging (MRI)

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I. What should go in this report?

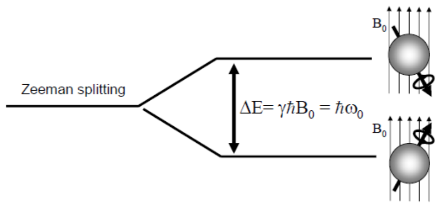
History of MRI, from NRM:

II. Review of Magnetic moments

Nuclei with an odd number or protons, an odd number of neutrons, or an odd number of protons and neutrons, have a magnetic moment denoted by $\vec{\mu}$. The spins for nuclei having an even number of protons and neutrons pair up and eliminate any magnetic moment. Examples of nuclei having magnetic moment are ^{1}H, ^{2}H, ^{3}He, ^{13}C, ^{17}O, ^{19}F, ^{23}Na, ^{31}P, etc. Example of nuclei which do not have magnetic moment are ^{4}He, ^{12}C, ^{16}O, etc. The magnetic moment magnitude is quantized so that

$$|\mu| = \frac{1}{2\pi}\gamma h\sqrt{I(I+1)}, I = 0, 1/2, 1, 3/2, 2, \cdots \tag{1}$$

where I is the spin quantum number which takes on either zero, half-integers, or integers.



Isotope	Spin I	% natural abundance of isotope	γ MHz/T	elemental abundance in body
^{1}H	1/2	99.985%	42.575	63%
^{2}H	1	0.015%	6.53	63%
^{13}C	1/2	1.108%	10.71	9.4%
^{14}N	1	99.63%	3.078	1.5%
^{15}N	1/2	0.37%	4.32	1.5%
^{17}O	5/2	0.037%	5.77	26%
^{19}F	1/2	100%	40.08	0%
^{23}Na	3/2	100%	11.27	0.041%
^{31}P	1/2	100%	17.25	0.24%

Fig. 1: From Noll's U. Michigan course notes

The net magnetization of a collection of nuclei is the sum of their individual moments:

$$\vec{M} = \sum_{i=1}^{N} \vec{\mu}_i. \tag{2}$$

When there is no external magnetic force, the net magnetization is zero because the thermal jostling of nuclei gives them a random orientation so that when a large number of nuclei are present, for every orientation there is typically one representing the opposite side. However, when there is an external magnetic moment $\vec{B}_0$, the nuclei line themselves up with this direction either along (spin up) or opposite the way $\vec{B}_0$ points (spin down). In this case, the component of μ along $\vec{B}_0$, μ_z is quantized

$$\mu_z = \frac{1}{2\pi}\gamma h m_I, \qquad m_I = -I, -I+1, \cdots, I-1, I. \tag{3}$$

This quantization implies that the angle of the magnetic moment with respect to $\vec{B}_0$ is also quantized

$$\cos(\theta) = \frac{\gamma h m_I}{\gamma h\sqrt{I(I+1)}} = \frac{m_I}{\sqrt{I(I+1)}}. \tag{4}$$

Thus, for a proton whose spin can be $\pm 1/2$, we have

$$\cos(\theta) = \frac{\pm 1/2}{\sqrt{1/2(3/2)}} = \frac{\pm 1}{\sqrt{3}}, \tag{5}$$

giving an angle of $\theta = 54.73^\circ$ in an up or a down spin fashion.

The angular momentum $\vec{J}$ of a particle is related to the magnetic moment $\vec{\mu}$ by the gyromagnetic ratio γ

$$\vec{\mu} = \gamma \vec{J}. \tag{6}$$

The gyromagnetic ratio depends on the particle. For Hydrogen ^1H it is $\gamma = 2.68$ x 10^8 rad/s/T. The value of γ is listed for some other types of material in TABLE XX.

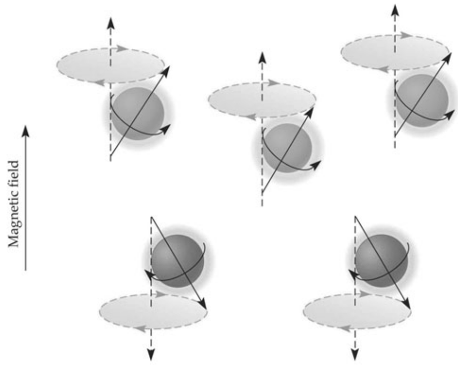


Fig. 2: [From MITCOGNET.edu]

III. Spin-Up vs Spin-down: The Boltzmann Distribution

Given a set of particles with a set of possible energy states, on what proportions do particles fill these energy states? This problem arises in various branches of science such as mathematics, chemistry, and physics, where a set of N particles are to fill in a set of states with energies $E_i, i = 1, 2, \cdots, P$. The Boltzmann distribution, known as the Gibbs measure in a more mathematical setting, states that the number of particles N_i in state with energy E_i is governed by

$$\frac{N_i}{N_j} = \frac{e^{-\frac{E_i}{kT}}}{e^{-\frac{E_j}{kT}}}, \tag{7}$$

where $k = 1.3806504\text{x}10^{-23}$ is the Boltzmann Constant and T is temperature. Therefore, we have

$$\frac{N_i}{N} = \frac{e^{-\frac{E_i}{kT}}}{\sum_{j=1}^{P} e^{-\frac{E_j}{kT}}}, \tag{8}$$

The Maxwell-Boltzmann distribution is a special case of the Boltzmann distribution which describes the velocities of particles of a gas.

The relevance of the Boltzmann distribution in MRI studies is the computation ofthe available number of detectable protons, i.e., the difference between the "spin-up" N^+ and "spin-down" N^- for a given total number of protons $N = N^+ + N^-$. Let the spin-up and spin-down energies be denoted as E^+ and E^-, respectively. Then,

$$\frac{N^-}{N^+} = \frac{e^{-\frac{E^-}{kT}}}{e^{-\frac{E^+}{kT}}} = e^{-\frac{E^- - E^+}{kT}} = e^{\frac{\Delta E}{kT}}, \tag{9}$$

where $\Delta E = E^+ - E^-$. This equation together with $N = N^+ + N^-$ determines N^+ and N^- as follows:

$$\frac{N}{N^+} - 1 = \frac{N - N^+}{N^+} = \frac{N^-}{N^+} = e^{\frac{\Delta E}{kT}}, \tag{10}$$

or

$$N^+ = \frac{N}{1 + e^{\frac{\Delta E}{kT}}}. \tag{11}$$

Then,

$$\frac{N^+ - N^-}{N} = \frac{N^+ - (N - N^+)}{N} = \frac{2N^+ - N}{N} = \frac{2}{1 + e^{\frac{\Delta E}{kT}}} - 1 = \frac{1 - e^{\frac{\Delta E}{kT}}}{1 + e^{\frac{\Delta E}{kT}}} \approx -\frac{\Delta E}{2kT} = \frac{\gamma h B_0}{4\pi kT}, \tag{12}$$

with the approximation holding when $\frac{\Delta E}{kT}$ is small. For example, in the case of a proton, under an external magnetic fields of 1.0 Tesla at room temperature of $T = 300$ leads to

$$\frac{N^+ - N^-}{N} = 3 \times 10^{-6}, \tag{13}$$

or 3 ppm (parts per million) protons available for MR imaging. That sounds like an awfully small number, except that in each gram or equivalently each milliliter of water there are 7x10^{19} protons so that a substantial number of protons are actively available for imaging.

The net magnetization is then

$$M = \sum_{i=1}^{N^+} \frac{1}{2\pi}\gamma h(\frac{+1}{2}) + \sum_{i=1}^{N^-} \frac{1}{2\pi}\gamma h(\frac{-1}{2}) = \frac{N^+ - N^-}{4\pi}\gamma h = \frac{\gamma^2 h^2}{(4\pi)^2 kT} B_0 N, \tag{14}$$

where the only parameters which can be changed are B_0 and T.

Symbol	Explanation of Notation
γ	gyromagnetic ratio: the ratio of its magnetic dipole moment to its angular momentum, radian per second per Tesla
$\gamma - line$	$\gamma/2\pi$
k	Botzmann Constant 1.38 x 10^{-23}
h	Plank's constant h=6.6x10^{-34} J-s
$h - line$	Plank's constant $/2\pi$

TABLE I: Summary of notation.

IV. The dynamics of a Magnetic Dipole

Let μ denote a magnetic dipole in an external magnetic field $\vec{B}_0$ which is aligned with the $z-$axis. Let $(\vec{i}, \vec{j}, \vec{k})$ be the unit vectors along the x, y, and z coordinates, respectively. Write

$$\vec{\mu} = \mu_x \vec{i} + \mu_y \vec{j} + \mu_z \vec{k}, \tag{15}$$

and $\vec{B}_0 = B_0 \vec{k}$. Then, the torque τ experienced by μ under the external magnetic field $\vec{B}_0$ is

$$\tau = \mu \times \vec{B}_0. \tag{16}$$

Now, the torque is equal to the change in angular momentum J, i.e., $\tau = \frac{dJ}{dt}$ and the angular momentum itself is related to the magnetic moment μ by $\mu = \gamma J$, so that

$$\frac{d\mu}{dt} = \gamma \mu \times \vec{B}_0. \tag{17}$$

The vector equation can now be considered as three scalar equations:

$$\begin{cases} \frac{d\mu_x}{dt} = \gamma B_0 \ \mu_y \\ \frac{d\mu_y}{dt} = -\gamma B_0 \ \mu_x \\ \frac{d\mu_z}{dt} = 0. \end{cases} \tag{18}$$

The first observation is that μ_z does not change. The second is that the components of μ in the $x - y$ plane are coupled. We can resolve this by differentiating the first equation a second time and substituting from the second equation. But first define $\omega_0 = \gamma B_0$.

$$\frac{d^2\mu_x}{dt^2} = -\omega_0^2 \mu_x, \tag{19}$$

and similarly

$$\frac{d^2\mu_y}{dt^2} = -\omega_0^2 \mu_y, \tag{20}$$

which are known differential equations with sinusoidal solutions of the form

$$\mu_x(t) = a\sin(\omega_0 t + \phi), \tag{21}$$

where the amplitude a and the phase ϕ are arbitrary. Using the first of Equations **??** gives

$$\mu_y(t) = \frac{1}{\omega_0}\frac{d\mu_x}{dt} = a\cos(\omega_0 t + \phi) \tag{22}$$

and

$$\mu_z(t) = b, \tag{23}$$

where b is an arbitrary constant which can be summarized as

$$\begin{cases} \mu_x(t) = a\sin(\omega_0 t + \phi), \\ \mu_y(t) = a\cos(\omega_0 t + \phi), \\ \mu_z(t) = b. \end{cases} \tag{24}$$

Now, at time 0 we have

$$\begin{cases} \mu_x(0) = a\sin(\phi), \\ \mu_y(0) = a\cos(\phi), \\ \mu_z(0) = b, \end{cases} \tag{25}$$

so that

$$\begin{cases} \mu_x(t) = a\sin(\omega_0 t)\cos(\phi) + a\cos(\omega_0 t)\sin(\phi), \\ \mu_y(t) = a\cos(\omega_0 t)\cos(\phi) - a\sin(\omega_0 t)\sin(\phi), \\ \mu_z(t) = b. \end{cases} \tag{26}$$

or in matrix form

$$\begin{bmatrix} \mu_x(t) \\ \mu_y(t) \\ \mu_z(t) \end{bmatrix} = \begin{bmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_x(0) \\ \mu_y(0) \\ \mu_z(0) \end{bmatrix}, \tag{27}$$

This proves that the magnetic moment of a particle (proton) has a constant z component, $\mu_z(0)$, while the x-y plane component revolves around the z-axis in a cone at a rate equal to $\omega_0 = \gamma B_0$. This movement is called the *Larmor* [1] *Precession* and the rate is called the *Larmor frequency.*

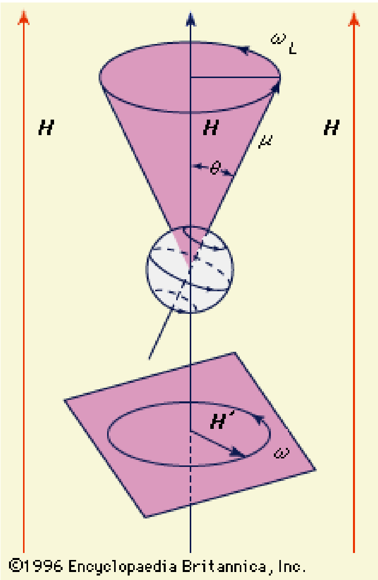


Fig. 3: Larmor precession [From Encyclopeda Britanica Inc.]

[1]Named after physicist and mathematician Sir Joseph Larmor (1857-1942) who made innovations in the understanding of electricity, dynamics, thermodynamics, and the electron theory of matter. [Wikipedia]

V. How do you get a proton excited?

Nuclei with magnetic moments respond to external magnetic fields, so if you want a nucleus to go the one you want it to you would happy to prod it along by applying yet a second magnetic filed, $\vec{B}_1(t)$. The key behind the NMR is *resonance*. Imagine a set of a dozen swings lined up in a row in a children's park. As the kids get on the swing, they typically go back and forth without any synchronization but with some fixed frequency that is characteristic of these identical swings. As some adults begin to push them, however, if the actions of the adults are synchronized, the swings also eventually get synchronized. Implicit in this is that adults naturally match they pushing to the swing's frequency. The matching of an effective outside force frequency to that of what it is being applied to is resonance. This is exactly what happens to protons under an outside magnetic field that can exactly the same frequency as the Larmor frequency. The outside magnetic field is simply an RF pulse of short duration, whose frequency is that of the Larmor frequency, for example 42.6 MHz for imaging protons under a 1.0 T magnet. The RF pulse can be written as

$$B_1(t) = 2B_1^e(t)\cos(\omega_0 t + \phi)\vec{i}, \tag{28}$$

which is along the x-axis of the subject (left to right). The envelop of the pulse $B_1^e(t)$ can be a rectangular pulse. It is convenient to think of this as two waves:

$$B_1(t) = B_1^e(t)[\cos(\omega_0 t+\phi)\vec{i}-\sin(\omega_0 t+\phi)\vec{j}]+B_1^e(t)[\cos(\omega_0 t+\phi)\vec{i}+\sin(\omega_0 t+\phi)\vec{j}], \tag{29}$$

which can be interpreted as two rotating waves one moving clockwise and one moving anti-clockwise.

$$B_1(t) = B_1^e(t)[e^{-i(\omega_0 t+\phi)}] + B_1^e(t)[e^{+i(\omega_0 t+\phi)}]. \tag{30}$$

The first wave is the only effective wave since it is along the precession. Quadrature RF transmitter coils generate this magnetic field as split in this way.

The effect of the external RF pulse is to get the spins to fall in common phase so that the net magnetization along the x-y plan is no longer zero. Rather, the x-y component rotates around the z-axis with the Larmor Frequency. The changes in this rotating net magnetization $\vec{M}$ can best be understood if we introduce a rotating frame $(\vec{\bar{i}}, \vec{\bar{j}}, \vec{\bar{k}})$ that rotates around the z-axis of the coordinate system $(\vec{i}, \vec{j}, \vec{k})$, defined by

$$\begin{bmatrix} \vec{\bar{i}} \\ \vec{\bar{j}} \\ \vec{\bar{k}} \end{bmatrix} = \begin{bmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}. \tag{31}$$

The derivatives of the new coordinate unit vectors are

$$\begin{bmatrix} \frac{d\vec{\bar{i}}}{dt} \\ \frac{d\vec{\bar{j}}}{dt} \\ \frac{d\vec{\bar{k}}}{dt} \end{bmatrix} = \begin{bmatrix} -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ -\cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = \begin{bmatrix} \vec{\bar{j}} \\ -\vec{\bar{i}} \\ 0 \end{bmatrix}. \tag{32}$$

Define the rotation vextor $\vec{\omega} = \omega_0 \vec{k}$. Then, we can write

$$\begin{bmatrix} \frac{d\vec{\bar{i}}}{dt} \\ \frac{d\vec{\bar{j}}}{dt} \\ \frac{d\vec{\bar{k}}}{dt} \end{bmatrix} = \begin{bmatrix} -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ -\cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = \begin{bmatrix} \vec{\bar{j}} \\ -\vec{\bar{i}} \\ 0 \end{bmatrix}. \tag{33}$$

Now, let the expression of the net magnetization in the original coordinate be

$$\vec{M} = M_x \vec{i} + M_y \vec{j} + M_z \vec{k}, \tag{34}$$

while the expression in the rotating coordinate system is

so that

$$[M_x \quad M_y \quad M_z] \;=\; [\bar{M}_x \quad \bar{M}_y \quad \bar{M}_z] \begin{bmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

or

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} \cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ \sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{M}_x \\ \bar{M}_y \\ \bar{M}_z \end{bmatrix} \tag{37}$$

Now, we can related the differential of M to (i) the change in M in the in the relative rotating system

$$\begin{aligned} \frac{d\vec{M}}{dt} = \frac{d\vec{\bar{M}}}{dt} &= \frac{d\bar{M}_x}{dt}\vec{\bar{i}} + \frac{d\bar{M}_y}{dt}\vec{\bar{j}} + \frac{d\bar{M}_z}{dt}\vec{\bar{k}} + \bar{M}_x\frac{d\vec{\bar{i}}}{dt} + \bar{M}_y\frac{d\vec{\bar{j}}}{dt} + \bar{M}_z\frac{d\vec{\bar{k}}}{dt} \\ &= \frac{\partial M_{rot}}{\partial t} + \end{aligned} \tag{38}$$

$$\frac{\partial M_{rot}}{\partial t} = \frac{d\bar{M}_x}{dt}\vec{\bar{i}} + \frac{d\bar{M}_y}{dt}\vec{\bar{j}} + \frac{d\bar{M}_z}{dt}\vec{\bar{k}} \tag{39}$$

VI. What is the MRI signal?

The precessing magnetization generates the signal we receive in MRI.

After the external magnetic pulse has been turned off the magnetic dipoles begin to align again with the external magnetic field B0. This process is called *relaxation* and is due to two distinct components. First, the so-called spin-lattice interaction where

Element	Symbol	Natural Abundance
Hydrogen	^{1}H	99.985
	^{2}H	0.015
Carbon	^{13}C	1.11
Nitrogen	^{14}N	99.63
	^{15}N	0.37
Sodium	^{23}Na	100
Phosphorus	^{31}P	100
Potassium	^{39}K	93.1
Calcium	^{43}Ca	0.145

Element	Biological Abundance*
Hydrogen (H)	0.63
Sodium (Na)	0.00041
Phosphorus (P)	0.0024
Carbon (C)	0.094
Oxygen (O)	0.26
Calcium (Ca)	0.0022
Nitrogen (N)	0.015

Fig. 4: Table

the dipoles are jostled by the thermal motion of the nearby molecules. The process is gradual where increasingly more dipoles return to align with the external magnetic field, the z-axis.

TABLE II: Signifiucant Milestones in the developments of MRI (QUOTED FROM u. mICHGICAN, Noll Notes

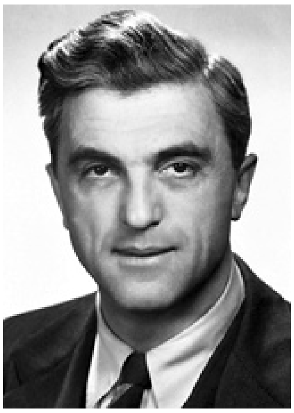
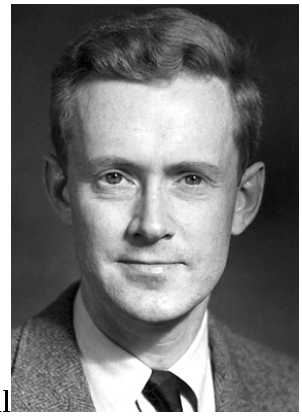
1946	MR phenomenon - Bloch and Purcell
1950	Spin echo signal discovered - Erwin Hahn
1952	Nobel Prize - Bloch and Purcell
1950-1970	NMR developed as analytical tool
1972	Computerized Tomography
1973	Backprojection MRI - Lauterbur
1975	Fourier Imaging - Ernst (phase and frequency encoding)
1977	MRI of the whole body - Raymond Damadian Echo-planar imaging (EPI) technique - Peter Mansfield
1980	MRI demonstrated - Edelstein
1986	Gradient Echo Imaging NMR Microscope
1988	Angiography - Dumoulin
1989	Echo-Planar Imaging (images at video rates = 30 ms / image)
1991	Nobel Prize - Ernst
1993	Functional MRI (fMRI)
1994	Hyperpolarized 129Xe Imaging
2003	Nobel Prize Lauterbur and Mansfield

VII. History of MRI/NRM

?

Highlight from Harnack:

The Nobel Prize in Physics in 1952 was given to Felix Bloch and E. M. Purcell, ”for their development of new methods for nuclear magnetic precision measurements and discoveries in connection therewith" The following is a bio from the Nobel prize web site.

Felix Bloch  Edward Purell

Felix Bloch was born in Zurich, Switzerland, on October 23, 1905, as the son of Gustav Bloch and Agnes Bloch (ne Mayer). From 1912 to 1918 he attended the public primary school and subsequently the "Gymnasium" of the Canton of Zurich, which he left in the fall of 1924 after having passed the "Matura", i.e. the final examination which entitled him to attend an institution of higher learning.

Planning originally to become an engineer, he entered directly the Federal Institute of Technology (Eidgenssische Technische Hochschule) in Zurich. After one year's study of engineering he decided instead to study physics, and changed therefore over to the Division of Mathematics and Physics at the same institution. During the following two years he attended, among others, courses given by Debye, Scherrer, Weyl, as well as Schrdinger, who taught at the same time at the University of Zurich and through whom he became acquainted, toward the end of this period, with the new wave mechanics. Bloch's interests had by that time turned toward theoretical physics. After Schrdinger left Zurich in the fall of 1927 he continued his studies with Heisenberg at the University of Leipzig, where he received his degree of Doctor of Philosophy in the summer of 1928 with a dissertation dealing with the quantum mechanics of electrons in crystals and developing the theory of metallic conduction. Various assistantships and fellowships, held in the following years, gave him the opportunity to work with Pauli, Kramers, Heisenberg, Bohr, and Fermi, and to further theoretical studies of the solid state as well as of the stopping power of charged particles.

Upon Hitler's ascent to power, Bloch left Germany in the spring of 1933, and a year later he accepted a position which was offered to him at Stanford University. The new environment in which he found himself in the United States helped toward the maturing of the wish he had had for some time to undertake also experimental research. Working with a very simple neutron source, it occurred to him that a direct proof for the magnetic moment of the free neutrons could be obtained through the observation of scattering in iron. In 1936, he published a paper in which the details of the phenomenon were worked out and in which it was pointed out that it would lead to the production and observation of polarized neutron beams. The further development of these ideas led him in 1939 to an experiment, carried out in collaboration with L.W. Alvarez at the Berkeley cyclotron, in which the magnetic moment of the neutron was determined with an accuracy of about one percent.

During the war years Dr. Bloch was also engaged in the early stages of the work on atomic energy at Stanford University and Los Alamos and later in counter-measures against radar at Harvard University. Through this latter work he became acquainted with the modern developments of electronics which, toward the end of the war, suggested to him, in conjunction with his earlier work on the magnetic moment of the neutron, a new approach toward the investigation of nuclear moments.

These investigations were begun immediately after his return to Stanford in the fall of 1945 and resulted shortly afterward in collaboration with W.W. Hansen and M.E. Packard in the new method of nuclear induction, a purely electromagnetic procedure for the study of nuclear moments in solids, liquids, or gases. A few weeks after the first successful experiments he received the news of the same discovery having been made independently and simultaneously by E.M. Purcell and his collaborators at Harvard.

Most of Bloch's work in the subsequent years has been devoted to investigations with the use of this new method. In particular, he was able, by combining it with the essential elements of his earlier work on the magnetic moment of the neutron, to remeasure this important quantity with great accuracy in collaboration with D. Nicodemus and H.H. Staub (1948). His more recent theoretical work has dealt primarily with problems which have arisen in conjunction with experiments carried out in his laboratory.

In 1954, Bloch took a leave of absence to serve for one year as the first Director General of CERN in Geneva. After his return to Stanford University he continued his investigations on nuclear magnetism, particularly in regard to the theory of relaxation. In view of new developments, a major part of his recent work deals with the theory of superconductivity and of other phenomena at low temperatures.

In 1961, he received an endowed Chair by his appointment as Max Stein Professor of Physics at Stanford University.

Prof. Bloch married in 1940 Dr. Lore Misch, a refugee from Germany and herself a physicist.